Math Diversion Problem 312

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With me, everything turns into mathematics.
— Rene Descartes
(P.S. I calculate; therefore I am.)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=4CZ3GFZ1ORA
Title: Can We Solve A Very Exponential Equation?|
            Problem 243
 Presenter: aplusbi
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1 The Problem

Given the relation

$$e^{i^z} = i\,, \tag{1}$$

find the values of z.

Jump down to the Solution Section, if you wish!

2 Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z, but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi\,, \tag{2}$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z.

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (2) is

$$\overline{z} = a - bi\,. \tag{3}$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2\,. \tag{4}$$

So, $z\overline{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \overline{z} = (a + bi) + (a - bi) = 2a\,. \tag{5}$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2}\,e^{i\theta} = (z\overline{z})^{1/2}e^{i\theta} = r\,e^{i\theta}\,, \tag{6}$$

where we can think of r as the length of the complex numbers z or $\overline{z}$.

$$r \equiv (z\overline{z})^{1/2} \quad \text{or} \quad r^2 = z\overline{z} = |\,z\,|^2\,. \tag{7}$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

3 The Solution

Before we do anything else, we need to place the number i into as general a form as we can. So,

$$e^{i^z} = i = e^{\frac{1}{2}\pi i + 2\pi i n} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{8}$$

Now we take the natural logarithm across the equation, which is the same as equating the exponents.

$$i^z = \tfrac{1}{2}\pi i + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{9}$$

Before we take another natural logarithm across the equation, let's replace the LHS this way:

$$i^z = \left(e^{\frac{1}{2}\pi i}\right)^z = e^{\frac{1}{2}\pi i z}\,. \tag{10}$$

Therefore, (9) becomes

$$e^{\frac{1}{2}\pi i z} = \tfrac{1}{2}\pi i + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{11}$$

Now we take one more natural logarithm:

$$\tfrac{1}{2}\pi i z = 2\pi i m + \ln\left(\tfrac{1}{2}\pi i + 2\pi i n\right) \quad \text{where} \quad m, n \in \mathbb{Z}\,. \tag{12}$$

Lastly, we divide through by $\frac{1}{2}\pi i$, to get

$$z = 4m - \frac{2i}{\pi}\,\ln\left(\tfrac{1}{2}\pi i + 2\pi i n\right) \quad \text{where} \quad m, n \in \mathbb{Z}\,. \tag{13}$$