

Math Diversion Problem 313

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When I give this talk to a physics audience, I
remove the quotes from my 'Theorem'.
— Brian Greene

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=GJbzsmccFtw>

Title: all solutions to $2^x - 3x - 1 = 0$ (transcendental equation)

Presenter: blackpenredpen

1 The Problem

Given the relation

$$2^x - 3x - 1 = 0, \tag{1}$$

find the values of x .

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B, \tag{2}$$

then

$$z = W(B), \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

The following is the 'Lambert W function base s '¹, or W_s , where s is a positive real number. Let's begin with the relation

$$xs^x = A, \tag{4}$$

¹This notation I invented myself.

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}. \quad (5)$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A), \quad (6)$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

3 The Solution

Let's begin by stating the trivial solution $x = 0$. But can we find any others?

My first instinct was to use the Lambert W function. But before we can invoke the Lambert W function, we have to set up the relation for it. To make appearances easier, I want to introduce a temporary variable k , defined by

$$3k = 3x + 1, \quad \text{so} \quad x = k - \frac{1}{3}. \quad (7)$$

With this, (1) becomes

$$2^{k-1/3} = 3k. \quad (8)$$

With a little algebra, this becomes

$$-k2^{-k} = -\frac{1}{3\sqrt[3]{2}}. \quad (9)$$

Applying the Lambert W function base 2, we get

$$-k = W_2\left(-\frac{1}{3\sqrt[3]{2}}\right) = \frac{W\left(-\frac{\ln 2}{3\sqrt[3]{2}}\right)}{\ln 2}. \quad (10)$$

Finally,

$$x = -\frac{1}{3} - \frac{W\left(-\frac{\ln 2}{3\sqrt[3]{2}}\right)}{\ln 2}. \quad (11)$$