# Math Diversion Problem 314

P. Reany

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> Thermodynamics is Nature's way of balancing
> entropy with enthalpy.
> — Rafael Jaramillo

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=0SDxMbrKh7E
Title:When Two Functions Are Tangent
 Presenter: SyberMath Shorts
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## 1 The Problem

Given the relation

$$e^x = \sqrt{ax}\,, \tag{1}$$

find the values of $x$.

## 2 The Preparation

I intend to use the Lambert $W$ function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on $B$ that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

The following is the 'Lambert $W$ function base $s$'[1], or $W_s$, where $s$ is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{4}$$

[1]This notation I invented myself.

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{5}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{6}$$

which is the usual Lambert $W$ function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

# 3 The Solution

Let's begin by squaring both sides:

$$(e^2)^x = ax\,. \tag{7}$$

With a little algebra, this becomes

$$-\frac{1}{a} = -x(e^2)^{-x}\,. \tag{8}$$

Hence, we can solve for $-x$:

$$-x = W_{e^2}\left(\frac{-1}{a}\right). \tag{9}$$

Lastly, we can solve for $x$:

$$x = -\frac{W\left(\frac{-1}{a}\ln e^2\right)}{\ln e^2} = -\frac{1}{2}W\left(\frac{-2}{a}\right). \tag{10}$$