Math Diversion Problem 316

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> Self-education is, I firmly believe, the only
> kind of education there is.
> — Isaac Asimov

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=mQvIXgfTtW4
Title: A very tricky Question from Stanford University
         Admission Exam
 Presenter: Super Academy
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1 The Problem

Given the relations

$$4^x - 4^y = 24\,, \tag{1a}$$

$$2^{x+y} = 35\,, \tag{1b}$$

find the values of $x - y$ for real x, y.

2 The Solution

Let's begin by squaring (1b):

$$4^{x+y} = 35^2 = 1225\,. \tag{2}$$

Now, let's multiply (1a) through by 4^y:

$$4^{x+y} - 4^{2y} = 24 \cdot 4^y\,. \tag{3}$$

From these last two equations, we can form a quadratic in 4^y:

$$(4^y)^2 + 24(4^y) - 1225 = 0\,, \tag{4}$$

which has solutions

$$4^y = -12 \pm 37\,. \tag{5}$$

From this, the real value of y requires the positive root, therefore,

$$4^y = 25\,. \tag{6}$$

Taking the square root of this, we get

$$2^y = 5\,, \tag{7}$$

with solution

$$y = \frac{\log 5}{\log 2}\,. \tag{8}$$

On taking the logarithm of (1b), we have that

$$x + y = \frac{\log 35}{\log 2} = \frac{\log 7 + \log 5}{\log 2}\,. \tag{9}$$

From these last two equations, we conclude that

$$x = \frac{\log 7}{\log 2}\,. \tag{10}$$

Finally,

$$x - y = \frac{\log 7 - \log 5}{\log 2} = \frac{\log\,(7/5)}{\log 2}\,. \tag{11}$$