Math Diversion Problem 317

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> The most dangerous phrase in the language is,
> 'We've always done it this way.'
> — Grace Hopper, computer pioneer

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=1hD6-_QPTBg
Title: A very tricky Question from Cambridge University
        Entrance Exam
 Presenter: Super Academy
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1 The Problem

Given the relation

$$a + 3125^a = 0\,, \tag{1}$$

find the values of a.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

Proof: Let $y = e^w$, then

$$W(e^w(w)) = W(we^w) = w = \ln y\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

3 The Solution

Before we can use the above lemma, we must first put the relation into an appropriate form. So, with some algebra,

$$-a = 3125^a\,, \tag{9}$$

or

$$-a\,3125^{-a} = 1\,. \tag{10}$$

We can now solve for $-a$:

$$-a = W_{3125}(1) = \frac{W(\ln 3125)}{\ln 3125}\,. \tag{11}$$

But $3125 = 5^5$, so

$$a = -\frac{W(5\ln 5)}{5\ln 5}\,. \tag{12}$$

If we want to be very general, we can write for the answer

$$a = -\frac{W_n(5\ln 5)}{5\ln 5} \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{13}$$

But for the principal value, which is (12), we can use the first lemma (4) to simplify:

$$a = -\frac{W(5\ln 5)}{5\ln 5} = -\frac{\ln 5}{5\ln 5} = -\frac{1}{5}\,. \tag{14}$$

[1] This notation I invented myself.