# Math Diversion Problem 319

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First things first...But not necessarily in that order.
— Doctor Who

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=-yo6-nOXeFI
Title: A tricky Question from Harvard University
      Admission Algebra Interview
 Presenter: Super Academy
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## 1 The Problem

Given the relation

$$\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}} = 10\,, \tag{1}$$

find the real values of $x$.

## 2 The Preparation

I'll be using a geometric series to help me solve this problem. In a geometric series, a number of terms are added together. Our interest will be on finite series. A generic series would look like this

$$G_n = \sum_{k=0}^{n} a_k\,, \tag{2}$$

where there is no stipulated relationships between the terms. What makes the geometric series special is 1) that the first term of the series is unity, and 2) that the ratio of a given term to its predecessor is a fixed value, which we will represent by the letter $r$. Thus, for this kind of series $a_k = r^k$, and,

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + r^3 + \cdots + r^n\,. \tag{3}$$

It's not hard to show that there's a closed form for $S_n$, given by

$$S_n = \frac{1 - r^{n+1}}{1 - r}\,. \tag{4}$$

# 3 The Solution

The Given relation can be recast into this form:

$$x^{(1/2)+(1/4)+(1/8)+(1/16)} = 10\,. \tag{5}$$

Now, the sum

$$\text{sum} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \tag{6}$$

is close to the form that we can match it to the formula (4), with $r = \frac{1}{2}$. However, the first term in this series isn't unity, though that's easy enough to fix. We can recast it![1]

$$\text{sum} = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) - 1 \tag{7}$$

Thus, using the formula, we get

$$\text{sum} = S_4 - 1 = \frac{1 - (\frac{1}{2})^5}{1 - \frac{1}{2}} - 1 \tag{8}$$

$$= \frac{1 - \frac{1}{32}}{\frac{1}{2}} - 1 \tag{9}$$

$$= \frac{15}{16}\,. \tag{10}$$

So, (5) becomes

$$x^{15/16} = 10\,. \tag{11}$$

Therefore,

$$x = 10^{16/15} = 10\sqrt[15]{10}\,. \tag{12}$$

[1]We don't have to use the geometric series, but I think it's good practice to do so here.