Math Diversion Problem 321

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> There's a remarkable discovery I've made by solving these
> 'olympiad'-style problems, which is that so many
> of them are just cleverly disguised quadratic
> or cubic equations.
> — The Author

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=c30AAa8EbQw
Title: Cambridge University Admission Aptitude Test
 Presenter: Super Academy
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1 The Problem

Given the relation

$$5^{\sin^2 4x} - 5^{\cos^2 4x} = 4\,, \tag{1}$$

find the real values of x.

2 The Solution

Clearly, we don't want a mixture of both sines and cosines, just one or the other. So, let's multiply through by $5^{\cos^2 4x}$.

$$5^{\sin^2 4x + \cos^2 4x} - 5^{2(\cos^2 4x)} = 4 \cdot 5^{\cos^2 4x}\,, \tag{2}$$

which simplifies to

$$5 - (5^{\cos^2 4x})^2 = 4 \cdot 5^{\cos^2 4x}\,. \tag{3}$$

But this is just a carefully disguised quadratic.

$$(5^{\cos^2 4x})^2 + 4 \cdot 5^{\cos^2 4x} - 5 = 0\,. \tag{4}$$

To simplify, let $y = 5^{\cos^2 4x}$, then

$$y^2 + 4y - 5 = 0\,, \tag{5}$$

or

$$(y+5)(y-1)=0\,, \tag{6}$$

with roots

$$y=1, \quad y=-5\,. \tag{7}$$

Case 1) $y=1$: then

$$5^{\cos^2 4x}=1\,. \tag{8}$$

For this to be true, then $\cos 4x=0$, therefore $4x=\pi/2\pm n\pi$ for $n\in\mathbb{Z}$, therefore,

$$x=\pi/8\pm n\pi/4 \quad \text{for} \quad n\in\mathbb{Z}\,. \tag{9}$$

If we restrict the domain from $0\leq x\leq\pi$ then there is the unique answer $x=\pi/8$.

Case 2) $y=-5$ has no real solutions.