Math Diversion Problem 322

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> The essence of mathematics is not to make simple things complicated, but to make complicated things simple.
> — S. Gudder

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=qNOrL5fBVVM
Title: High School Math Tournament Disrupted By Advanced Algebra!
 Presenter: Super Academy
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1 The Problem

Given the relation

$$(x-3)^{\sqrt{x-3}} = 3\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

Proof: Let $y = e^w$, then

$$W(e^w(w)) = W(we^w) = w = \ln y\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

3 The Solution

Before we can use the above lemma, we must first put the relation into an appropriate form. So, let's start by taking the square root of both sides.

$$(\sqrt{x-3})^{\sqrt{x-3}} = \pm\sqrt{3}\,. \tag{9}$$

Now we take the logarithm.

$$\sqrt{x-3}\ln\sqrt{x-3} = \pm\tfrac{1}{2}\ln 3\,. \tag{10}$$

Next, we use the lemma in (7):

$$\ln\sqrt{x-3} = W(\pm\tfrac{1}{2}\ln 3)\,. \tag{11}$$

A little more algebra and we get

$$x = e^{2W(\pm\frac{1}{2}\ln 3)} + 3\,, \tag{12}$$

where the real solution uses the plus sign.

[1]This notation I invented myself.