Math Diversion Problem 324

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> A mathematician is a machine for turning
> coffee into theorems.
> — Alfred Renyi

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=C-6YhHScNvI
Title: Oxford University Entrance Interview
 Presenter: Super Academy
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1 The Problem

Given the relations

$$10x^2 + 10y^2 = 29xy\,, \tag{1a}$$
$$x^2 - y^2 = 21\,, \tag{1b}$$

find the values of $x + y$ for real x, y.

2 The Solution

First, I want to change the Given relations to the following form:

$$10x^2 + 10y^2 = 29xy\,, \tag{2a}$$
$$10x^2 - 10y^2 = 210\,. \tag{2b}$$

If we add them together and then subtract them, we get:

$$20x^2 = 29xy + 210\,, \tag{3a}$$
$$20y^2 = 29xy - 210\,. \tag{3b}$$

Next, we multiply these together to get a quadratic in xy.

$$400(xy)^2 = 29^2(xy)^2 - 210^2\,, \tag{4}$$

which gives us

$$xy = \pm 10\,. \tag{5}$$

However, from (2a) and keeping to the requirement that x, y have real values, we must insist that x and y have the same sign, forcing us to disregard the minus sign in (5). Thus,

$$xy = 10\,. \tag{6}$$

Next, we can take this value and put it into (3a), to give us

$$10x^2 + 10y^2 = 290\,, \tag{7a}$$

$$10x^2 - 10y^2 = 210\,. \tag{7b}$$

Thus, adding them, we get

$$20x^2 = 210 + 290 = 500\,, \tag{8}$$

yielding

$$x = \pm 5\,. \tag{9}$$

From (6), we can figure out y by inspection:

$$y = \pm 2\,. \tag{10}$$

Anyway, since we are asked to solve for $x + y$, we have that

$$x + y = \pm 7\,. \tag{11}$$