Math Diversion Problem 325

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January 27, 2025

The greatest killer of creativity is interruption.
— John Cleese

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=wzMwFeIE-uU
Title: Harvard University Admission Interview Tricks
          You Should Know!
 Presenter: Super Academy
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1 The Problem

Given the relation

$$(\log_3 2)^x + (\log_2 3)^x = 6\,, \tag{1}$$

find the values of x.

2 The Preparation

Lemma: Fundamental Rule of Logarithmic Conversion (base a to base 10):

$$\log_a x = \frac{\log x}{\log a}\,. \tag{2}$$

3 The Solution

We can start by converting to a common base, such as base 10. Applying the above lemma, then

$$\left(\frac{\log 2}{\log 3}\right)^x + \left(\frac{\log 3}{\log 2}\right)^x = 6\,, \tag{3}$$

So, what is this? It's one of those cleverly disguised quadratic equations. Let

$$y \equiv \frac{\log 2}{\log 3}\,. \tag{4}$$

Then (3) becomes

$$y^x + y^{-x} = 6\,. \tag{5}$$

On multiplying this through by y^x, we have that

$$y^{2x} + 1 = 6y^x\,. \tag{6}$$

That this is a quadratic can be demonstrated easily.

$$\bigl(y^x\bigr)^2 - 6\bigl(y^x\bigr) + 1 = 0\,. \tag{7}$$

Therefore

$$y^x = \frac{6 \pm \sqrt{36 - 4(1)(1)}}{2} = 3 \pm 2\sqrt{2}\,. \tag{8}$$

Hence

$$x \log y = \log(3 \pm 2\sqrt{2})\,. \tag{9}$$

And finally

$$x = \frac{\log\,(3 \pm 2\sqrt{2})}{\log\,[(\log 2)/(\log 3)]}\,, \tag{10}$$

for the real values of x.