Math Diversion Problem 327

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You will never plough a field if you only turn
it over in your mind.
— Irish Proverb

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Tz4eP8l34O8
Title: Surprising Result from a Complex Expression | Problem 20
 Presenter: aplusbi
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1 The Problem

Given the relation

$$\phi = \frac{1 + \cos\theta + i\sin\theta}{1 + \cos\theta - i\sin\theta}\,, \tag{1}$$

simplify ϕ.

Jump down to the Solution Section, if you wish!

2 Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z, but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi\,, \tag{2}$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z.

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (2) is

$$\overline{z} = a - bi\,. \tag{3}$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2\,. \tag{4}$$

So, $z\overline{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \overline{z} = (a + bi) + (a - bi) = 2a\,. \tag{5}$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2}e^{i\theta} = (z\overline{z})^{1/2}e^{i\theta} = r\,e^{i\theta}\,, \tag{6}$$

where we can think of r as the length of the complex numbers z or $\overline{z}$.

$$r \equiv (z\overline{z})^{1/2} \quad \text{or} \quad r^2 = z\overline{z} = |\,z\,|^2\,. \tag{7}$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

3 The Solution

My first inclination was to use Euler's Formula to convert the Given relation to:

$$\phi = \frac{1 + e^{i\theta}}{1 + e^{-i\theta}}\,. \tag{8}$$

Then I tried to multiply numerator and denominator by the conjugate of the denominator, but that was a rabbit hole I decided to forsake. So, I tried something else, namely, clear of fractions.

$$\phi(1 + e^{-i\theta}) = 1 + e^{i\theta}\,. \tag{9}$$

Then, I multiplied through by $e^{i\theta}$ — why not?

$$\phi(e^{i\theta} + 1) = e^{i\theta} + e^{2i\theta}\,. \tag{10}$$

Now, notice that $e^{i\theta} + 1$ on the LHS of this last equation is the same as $1 + e^{i\theta}$ of the RHS (9), so, by use of the transitive property, the last equation becomes:

$$\phi^2(1 + e^{-i\theta}) = e^{i\theta} + e^{2i\theta} = e^{i\theta}(1 + e^{i\theta})\,. \tag{11}$$

Now, let's divide through by $1 + e^{-i\theta}$:

$$\phi^2 = e^{i\theta} \frac{1 + e^{i\theta}}{1 + e^{-i\theta}} = e^{i\theta} \phi \,. \tag{12}$$

Thus, if $\phi \neq 0$ (i.e., $1 + e^{i\theta} \neq 0$), we can divide through by ϕ, to get

$$\phi = e^{i\theta} \,. \tag{13}$$

Note: This answer is close to, but not quite the same as, what WolframAlpha gets. It gets that $1 + e^{-i\theta} \neq 0$, which has to be true from the get-go.