

Math Diversion Problem 331

P. Reany

January 27, 2025

Self-education is, I firmly believe, the only
kind of education there is.
— Isaac Asimov

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=R8rLyzJ9IK4>
Title: How to solve this?
Presenter: Super Academy

1 The Problem

Given the relation

$$x^{x^8} = 8, \tag{1}$$

find the real, positive values of x .

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B, \tag{2}$$

then

$$z = W(B), \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the following lemma:

$$W(y \ln y) = \ln y, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

Proof: Let $y = e^w$, then

$$W(e^w(w)) = W(we^w) = w = \ln y. \tag{5}$$

3 The Solution

I can think of two ways to solve this problem: The first is to perform a change of variable, making $x = 8^\alpha$, and taking it from there. The second is to use the Lambert W function, which is the route I'll take here.

So, let's begin by taking the natural logarithm across the Given relation, to get

$$x^8 \ln x = \ln 8. \tag{6}$$

Next, multiply through by 8:

$$8x^8 \ln x = 8 \ln 8, \tag{7}$$

or

$$x^8 \ln x^8 = 8 \ln 8. \tag{8}$$

Now, take the Lambert W function across this equation, to get

$$\ln x^8 = \ln 8, \tag{9}$$

where we used the above lemma. From this, we get that

$$x^8 = 8. \tag{10}$$

And finally,

$$x = 8^{1/8}. \tag{11}$$