Math Diversion Problem 336

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Easy to criticize, more difficult to be correct.
— Charlie Chan

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=PrQuiomB2oI
Title: An Interesting Cubic Equation | Problem 448
 Presenter: aplusbi
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1 The Problem

Given the relation

$$z^3 - z = 10i\,, \tag{1}$$

find the values of z.

2 The Preparation

Some factoring:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)\,, \tag{2a}$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)\,. \tag{2b}$$

Some facts of complex numbers $z, \overline{z}$, where $z = a+bi$, $\overline{z} = a-bi$, and $r^2 = a^2+b^2$:

$$z + \overline{z} = 2a\,, \tag{3a}$$

$$z - \overline{z} = 2bi\,, \tag{3b}$$

$$z\overline{z} = r^2\,, \tag{3c}$$

$$z^2 + \overline{z}^2 = (z + \overline{z})^2 - 2z\overline{z} = 4a^2 - 2r^2\,, \tag{3d}$$

$$z^2 - \overline{z}^2 = (z - \overline{z})(z + \overline{z}) = 4abi\,, \tag{3e}$$

$$\begin{aligned} z^3 + \overline{z}^3 &= (z + \overline{z})(z^2 - z\overline{z} + \overline{z}^2) = (z + \overline{z})[(z^2 + \overline{z}^2) - z\overline{z}] \\ &= (2a)[(4a^2 - 2r^2) - r^2] = (2a)(4a^2 - 3r^2)\,, \end{aligned} \tag{3f}$$

$$z^3 - \overline{z}^3 = (z - \overline{z})(z^2 + z\overline{z} + \overline{z}^2) = (2bi)(4a^2 - r^2)\,. \tag{3g}$$

3 The Solution

First, I want to add to the Given relation its complex conjugate.

$$z^3 - z = 10i\,, \tag{4a}$$
$$\overline{z}^3 - \overline{z} = -10i\,, \tag{4b}$$

Now, let's take the sum of these two relations, to get

$$z^3 + \overline{z}^3 - (z + \overline{z}) = 0\,. \tag{5}$$

But from the Preparation relations above, we can factor this to

$$(z + \overline{z})(z^2 - z\overline{z} + \overline{z}^2 - 1) = 0\,. \tag{6}$$

This brings us to two cases, depending on whether $z + \overline{z} = 0$ or not.

Case 1: $z + \overline{z} = 0$.

Hence

$$a = 0\,. \tag{7}$$

Therefore, (1) must be satisfied by a pure imaginary number $z = bi$.

$$(bi)^3 - (bi) = 10i\,, \tag{8}$$

which becomes

$$-b^3 - b = 10\,, \tag{9}$$

which has the real solution

$$b = -2\,. \tag{10}$$

So, we've found one of the three solutions to (1), namely,

$$z = -2i\,. \tag{11}$$

Case 2: $z + \overline{z} \neq 0$.

Hence

$$z^2 - z\overline{z} + \overline{z}^2 - 1 = 0\,, \tag{12}$$

or

$$4a^2 - 3r^2 - 1 = 0\,, \tag{13}$$

which can be further reduced to

$$a^2 - 3b^2 - 1 = 0\,. \tag{14}$$

Remember that this last relation came from us taking the sum of the equations (4a) and (4b). This time, let's take their difference.

$$z^3 - \overline{z}^3 - (z - \overline{z}) = 20i\,. \tag{15}$$

But from the Preparation relations above, we can partially factor this to

$$(z - \overline{z})[(z^2 + z\overline{z} + \overline{z}^2) - 1] = 20i\,. \tag{16}$$

After we do the standard substitutions and simplifications (using (13)), we get

$$br^2 = 5\,, \tag{17}$$

or

$$b(a^2 + b^2) = 5\,. \tag{18}$$

Now we can use (14) with this last relation to eliminate a^2, leaving

$$4b^3 + b = 5\,, \tag{19}$$

which has real root $b = 1$. Using (18), we can calculate a,

$$a = \pm 2\,. \tag{20}$$

Thus, the last two solutions to z are

$$z = \pm 2 + i\,. \tag{21}$$

Comment: I would not call the solution I've presented as elegant. In fact, it's quite tedious, but at least it's straightforward.