

Math Diversion Problem 337

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A clue is anything that doesn't happen
the way it oughtta happen.
— Harry Orwell,
TV show *Harry O*

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=tKjykRzjRCc>
Title: A Radical Equation | Problem 416
Presenter: aplusbi

1 The Problem

Given the relation

$$z\sqrt{z} = -2 + 2i, \quad (1)$$

find the values of z .

Note: Skip down to the solution, if you like.

2 Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z , but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi, \quad (2)$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z .

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (2) is

$$\bar{z} = a - bi. \quad (3)$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2. \quad (4)$$

So, $z\bar{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \bar{z} = (a + bi) + (a - bi) = 2a. \quad (5)$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2}e^{i\theta} = (z\bar{z})^{1/2}e^{i\theta} = re^{i\theta}, \quad (6)$$

where we can think of r as the length of the complex numbers z or \bar{z} .

$$r \equiv (z\bar{z})^{1/2} \quad \text{or} \quad r^2 = z\bar{z} = |z|^2. \quad (7)$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

3 The Solution

First, I want to recast the Given relation to the form

$$z^{3/2} = -2 + 2i, \quad (8)$$

which seems correct and WolframAlpha gives the same answers to both of them, namely,

$$z = 2i; -2\sqrt[6]{-1} = -2e^{i\pi/6}. \quad (9)$$

Now, the RHS of (8) can be put into a convenient polar form as

$$z^{3/2} = -2 + 2i = 2\sqrt{2} \frac{-1 + i}{\sqrt{2}} = 2^{3/2} e^{3i\pi/4} e^{2i\pi k} \quad \text{where } k \in \mathbb{Z}. \quad (10)$$

The factor $e^{2i\pi k}$, which is unity, was inserted in anticipation of taking the 2/3th root. So, let's do take the 2/3th root across the equation, yet retain only those

values of k that produce unique values, namely $k = 0, 1, 2$. However, the values of k that produce roots to the original equation will have to be checked.

$$z = 2 e^{i\pi/2} e^{i\pi k/3} \quad \text{where } k \in \{0, 1, 2\}. \quad (11)$$

For $k = 0$,

$$z = 2 e^{i\pi/2} = 2i. \quad (12)$$

For $k = 1$,

$$z = 2 e^{i\pi/2} e^{2i\pi/3} = 2 e^{i\pi(\frac{1}{2} + \frac{2}{3})} = 2 e^{7i\pi/6} = 2 e^{i\pi} e^{i\pi/6} = -2 e^{i\pi/6}. \quad (13)$$

At this point, we already have the two solutions proposed by WolframAlpha. For $k = 2$,

$$z = 2 e^{i\pi/2} e^{4i\pi/3} = 2 e^{i\pi(\frac{1}{2} + \frac{4}{3})} = 2 e^{11i\pi/6} = -2 e^{5i\pi/6}. \quad (14)$$

Anyway, when I take $2 e^{11i\pi/6}$ to the $\frac{3}{2}$ power, I get $-2 + 2i$, as is correct.