Math Diversion Problem 344

P. Reany

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There are five fundamental operations in mathematics:
Addition, subtraction, multiplication, division,
and modular forms.
— Martin Eichler

This problem is #8 on the webpage
https://www.basic-mathematics.com/hard-word-problems-in-algebra.html

1 The Problem

Word Problem:
One ounce of solution X contains only ingredients a and b in a ratio of 2:3. One ounce of solution Y contains only ingredients a and b in a ratio of 1:2. If solution Z is created by mixing solutions X and Y in a ratio of 3:11, then 2520 ounces of solution Z contains how many ounces of a?

If you're interested in more word problems, see my many articles on solving algebra word problems at:

`https://www.advancedmath.org/Math/AlgebraWordProblems.html`

(Skip down to the solution, if you like.)

2 Problem-Solving Aids (Heuristics)

One way to get really confused or even lost in solving a word problem is to do too much in a single step. The fix to this is to slow down and be very intentional about everything you write down. Develop the equations slowly. This philosophy should be apparent in the manner of the development of the heuristics and the problem solving technique employed below.

When I approach a word problem, I have a set of heuristics to guide me through the process:

1. If there are any totals or parts of a total lying around, put them into an equation (or into an inequality, if appropriate). Then, note that the following rule generates an equation: *Every total is equal to the sum of all its parts.*

$$\text{Total} = \sum_i \text{Parts}_i \,. \tag{1}$$

2. Is there some invariant Inv evidently holding from the initial state to the final state of a before-and-after process? If so, write

$$\text{Inv}_{\text{before}} = \text{Inv}_{\text{after}} \,. \tag{2}$$

And that's another equation to work with. For example, say we have a beaker containing 50 ml of salt solution (in water), to which we add 10 ml of water. Now, although the amount of water in the beaker has not remained invariant throughout this procedure, the amount of salt has. So, you can write an equation out of that invariance!

3. Is there a common or problem-specific formula to use? Such as from physics, chemistry, etc., or from mathematics, like from geometry or from number theory, such as for the summation of a series or for a weighted average of a set of numbers, or the greatest common factor or least common multiple of two or more numbers, and so on. For example, the area of a triangle $A = \frac{1}{2}bh$ is an equation. Furthermore, is there a problem-specific relationship given in the problem, such "the base of the triangle is one-third its height." That's an equation!

4. Is there a proportion given? A proportion is the stated equality of two ratios, such as

$$\frac{A}{B} = \frac{C}{D} \,. \tag{3}$$

5. Are there one or more linear or quadratic equations given? If so, write them down.

When we've collected as many equations as we have unknowns, we should be ready to solve the system simultaneously. (However, no one of these equations should be derivable from the others of the system.)

3 The Solution

Restatement of the problem: One ounce of solution X contains only ingredients a and b in a ratio of 2:3. One ounce of solution Y contains only ingredients a and b in a ratio of 1:2. If solution Z is created by mixing solutions X and Y in a ratio of 3:11, then 2520 ounces of solution Z contains how many ounces of a?

Note: What the problem calls 'ingredients', I will refer to as 'substances.'

The first thing is to create a graphic that contains the relevant information from this problem (see Figure below).

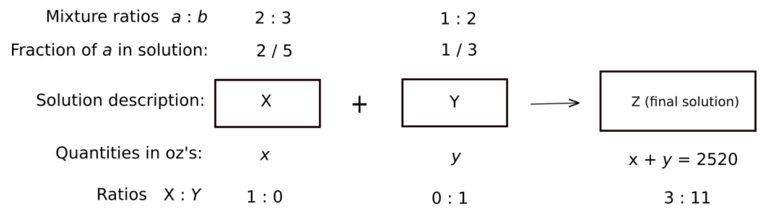

Figure 1. We need to solve for the amount of component a in the 2520 oz of mixture Z. To go from the ratio $a : b$ to the fractional equivalent, make the fraction $a/(a+b)$. The reason this works is because the mixtures contain only substances a and b. Let x, y represent the amounts of solution contributed from X,Y, respectively, to form the substance Z.

Question 1: What does that bottom ratios line mean in the figure above? Ans: Well, I needed to get the information into the figure that the ratio of X to Y in Z is 3 : 11, but I didn't want to leave the corresponding entries under X and Y blank. So, the ratios 1 : 0 and 0 : 1 are sort of definitions, meaning that X is all X and no Y, and vice versa. Or, think of them as placeholders.

Question 2: Are there any relevant totals in this problem? Yes! The total amount of substance a that ends up in solution Z is the sum of that which came from solution X plus that which came from solution Y.

$$T = (\text{total } a \text{ in Z}) = (\text{contribution } a \text{ from X}) + (\text{contribution } a \text{ from Y}) \tag{4a}$$

$$= (\text{fractional amount of } a \text{ in X})x + (\text{fractional amount of } a \text{ in Y})y \tag{4b}$$

$$T = \frac{2}{5}x + \frac{1}{3}y\,. \tag{4c}$$

So, once we know both x and y, we can solve for T.

To solve for x and y, we have two equations to work with:

$$x + y = 2520\,, \tag{5a}$$

$$\frac{x}{y} = \frac{3}{11}\,. \tag{5b}$$

Equation (5a) indicates our second total of the problem, and (5b) represents a proportionality. Solving (5a) and (5b) as a system, we get

$$x = 540\,, \tag{6a}$$

$$y = 1980\,. \tag{6b}$$

This gives us $T = 876$ in ounces.