Math Diversion Problem 345

P. Reany

January 27, 2025

Mathematics compares the most diverse phenomena and discovers
the secret analogies that unite them.
— Joseph Fourier

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=rEpON1AmV_o
Title: A Silently Trigonometric Equation
    | Problem 296
 Presenter: aplusbi
```

1 The Problem

Given the relation

$$\cos z + \sin z = i\,, \tag{1}$$

find the values for z.

Note: Skip down to the solution, if you like.

2 Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z, but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi\,, \tag{2}$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z.

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (2) is

$$\overline{z} = a - bi\,. \tag{3}$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2\,. \tag{4}$$

So, $z\overline{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \overline{z} = (a + bi) + (a - bi) = 2a\,. \tag{5}$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2}e^{i\theta} = (z\overline{z})^{1/2}e^{i\theta} = r\,e^{i\theta}\,, \tag{6}$$

where we can think of r as the length of the complex numbers z or $\overline{z}$.

$$r \equiv (z\overline{z})^{1/2} \quad \text{or} \quad r^2 = z\overline{z} = |\,z\,|^2\,. \tag{7}$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

3 The Solution

We'll be using these five identities:

$$(1 + i)^2 = 2i\,, \tag{8a}$$

$$(1 - i)^2 = -2i\,, \tag{8b}$$

$$(1 + i)(1 - i) = 2\,, \tag{8c}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}\,, \tag{8d}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}\,. \tag{8e}$$

Using these last two first, we get from (1)

$$i = \cos z + \sin z = \tfrac{1}{2}e^{iz}\Big(1 + \frac{1}{i}\Big) + \tfrac{1}{2}e^{-iz}\Big(1 - \frac{1}{i}\Big) \tag{9a}$$

$$= \tfrac{1}{2}e^{iz}(1 - i) + \tfrac{1}{2}e^{-iz}(1 + i)\,. \tag{9b}$$

Now, multiply through by $e^{iz}(1+i)$:

$$e^{iz}(-1+i) = e^{2iz} + i\,. \qquad (10)$$

Next, let $X = e^{iz}$ and then form a quadratic:

$$X^2 + (1-i)X + i = 0\,, \qquad (11)$$

which has roots[1]

$$X = \pm\Big(\frac{1}{2} - \frac{i}{2}\Big)(\mp 1 + \sqrt{3})\,. \qquad (12)$$

On taking the logarithm of $X = e^{iz}$ we get,

$$iz = \ln\,[\pm\Big(\frac{1}{2} - \frac{i}{2}\Big)(\mp 1 + \sqrt{3})] + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,. \qquad (13)$$

And finally,

$$z = -i\,\ln\,[\pm\Big(\frac{1}{2} - \frac{i}{2}\Big)(\mp 1 + \sqrt{3})] + 2\pi n \quad \text{where} \quad n \in \mathbb{Z}\,. \qquad (14)$$

WolframAlpha gives the roots as

$$z = 2\pi n \pm 2i\,\tanh^{-1}\big[\Big(\frac{1}{2} + \frac{i}{2}\Big)(\sqrt{3} \mp 1)\big] \quad \text{where} \quad n \in \mathbb{Z}\,. \qquad (15)$$

[1] Roots provided by WolframAlpha.