Math Diversion Problem 347

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> The ignorant always loud in argument.
> — Charlie Chan

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=11-n0gqmXwk
Title: A Nice Nonic Equation
    | Problem 408
 Presenter: aplusbi
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1 The Problem

Given the relation

$$(z+1)^{10} = z^{10}\,, \tag{1}$$

find the values for z.

Note: Skip down to the solution, if you like.

2 The Solution

Before we get too deep into the solution, let's take the time to notice that the z^{10} terms in (1) will cancel out, leaving us with a ninth-degree equation, and that implies only nine solutions to solve for.

One of my favorite tricks to use in situations like this, is to find a point half way between z and $z+1$, and call it w. So,

$$w = z + \tfrac{1}{2}, \quad z+1 = w + \tfrac{1}{2}, \quad z = w - \tfrac{1}{2}\,. \tag{2}$$

Therefore, (1) becomes

$$(w + \tfrac{1}{2})^{10} = (w - \tfrac{1}{2})^{10}\,. \tag{3}$$

We can rewrite this last equation to

$$(w + \tfrac{1}{2})^{10} - (w - \tfrac{1}{2})^{10} = 0\,. \tag{4}$$

But this is the difference of two squares:

$$\big((w+\tfrac{1}{2})^5-(w-\tfrac{1}{2})^5\big)\big((w+\tfrac{1}{2})^5+(w-\tfrac{1}{2})^5\big)=0\,. \tag{5}$$

WolframAlpha tells me that these factors expand as

$$\text{First factor}\quad (w+\tfrac{1}{2})^5-(w-\tfrac{1}{2})^5=5w^4+\frac{5w^2}{2}=\frac{1}{16}\,, \tag{6a}$$

$$\text{Second factor}\quad (w+\tfrac{1}{2})^5+(w-\tfrac{1}{2})^5=2w^5+5w^3+\frac{5w}{8}\,. \tag{6b}$$

When we set the first factor equal to zero, we get the four roots

$$w=\pm\left[\frac{i}{2}\sqrt{1-\frac{2}{\sqrt{5}}}\right]\,, \tag{7a}$$

$$w=\pm\left[\frac{i}{2}\sqrt{1+\frac{2}{\sqrt{5}}}\right]\,. \tag{7b}$$

And when we set the second factor equal to zero, we get the five roots

$$w=0\,, \tag{8a}$$

$$w=\pm\left[\frac{i}{2}\sqrt{1-\frac{2}{\sqrt{5}}}\right]\,, \tag{8b}$$

$$w=\pm\left[\frac{i}{2}\sqrt{1+\frac{2}{\sqrt{5}}}\right]\,. \tag{8c}$$

Thus, for z we get the nine roots:

$$z=\pm\left[\frac{i}{2}\sqrt{1-\frac{2}{\sqrt{5}}}\right]-\frac{1}{2}\,,\quad \pm\left[\frac{i}{2}\sqrt{1+\frac{2}{\sqrt{5}}}\right]-\frac{1}{2}\,. \tag{9}$$

And

$$z=-\frac{1}{2}\,, \tag{10a}$$

$$z=\pm\left[\frac{i}{2}\sqrt{1-\frac{2}{\sqrt{5}}}\right]-\frac{1}{2}\,, \tag{10b}$$

$$z=\pm\left[\frac{i}{2}\sqrt{1+\frac{2}{\sqrt{5}}}\right]-\frac{1}{2}\,. \tag{10c}$$

However, when I give WolframAlpha Eq. (1) to solve, it returns only five solutions.