

Math Diversion Problem 348

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It is clear that the chief end of mathematical study
must be to make the students think.
— John Wesley Young
(So think!)

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=zoytksx8dPE>
Title: A Nice Problem With Reciprocals
| Problem 466
Presenter: aplusbi

1 The Problem

Given the relation

$$z^2 - z + 1 = 0, \tag{1}$$

find the values of

$$z^5 + \frac{1}{z^5}, \tag{2}$$

Note: Skip down to the solution, if you like.

2 The Solution

One way to solve this problem is to solve for the roots to (1), and then plug those into the Find expression (2). But there's another way in which we don't need to solve for z at all. The Given relation will help us to do this. It can be rewritten as

$$z + \frac{1}{z} = 1. \tag{3}$$

Let

$$\phi \equiv z + \frac{1}{z}. \tag{4}$$

If we take ϕ^5 , we will get the terms $z^5, \frac{1}{z^5}$, but we will get other terms, including cubic ones. Let's deal with those now.

$$1 = \phi^3 = \left(z + \frac{1}{z}\right)^3 = z^3 + 3z + 3\frac{1}{z} + \frac{1}{z^3}. \quad (5)$$

But this becomes

$$1 = z^3 + \frac{1}{z^3} + 3, \quad (6)$$

or

$$z^3 + \frac{1}{z^3} = -2. \quad (7)$$

Now we're ready to take ϕ^5 and simplify (I use Pascal's Triangle):

$$\begin{aligned} 1 = \phi^5 &= \left(z + \frac{1}{z}\right)^5 = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right) \\ &= \left(z^5 + \frac{1}{z^5}\right) + 5(-2) + 10(1), \end{aligned} \quad (8)$$

which boils down to

$$z^5 + \frac{1}{z^5} = 1. \quad (9)$$