Math Diversion Problem 349

P. Reany

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> I feel that a visual representation of the Dirac algebra is of great
> benefit, because it can provide an additional insight that
> is not easily expressed with words or equations.
> — David M. Goodmanson
> ['A graphical representation of the Dirac algebra',
> American J. Phys., Vol. 64, No. 7,
> July 1996, p. 870.]

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=RQYwPdz9RWI
Title: Simple trigonometric equation
 Presenter: Tambuwal Maths Class
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1 The Problem

Given the relation

$$\sin x = 4^{-\sin x}\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.
To continue, I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

A lemma I'll need is the following: If

$$y\ln y = B\,, \tag{7}$$

then

$$\ln y = W(y\ln y) = W(B)\,. \tag{8}$$

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3 The Solution

There are other ways to solve this problem than by using the Lambert W function, but this method has a generality to it.

Let's begin by bringing the two sides together as factors:

$$(\sin x)4^{\sin x} = 1\,. \tag{9}$$

Next, we take the Lambert W function base 4 across the equation:

$$\sin x = W_4(1) = \frac{W(1\cdot\ln 4)}{\ln 4} = \frac{W(2\ln 2)}{2\ln 2} = \frac{\ln 2}{2\ln 2} = \frac{1}{2}\,, \tag{10}$$

where we used the second lemma this time. Hence,

$$x = \begin{cases} \dfrac{\pi}{6} + 2\pi n & \text{for} \quad n \in \mathbb{Z}\,, \\ \dfrac{5\pi}{6} + 2\pi m & \text{for} \quad m \in \mathbb{Z}\,. \end{cases} \tag{11}$$