Math Diversion Problem 351

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After a time, you may find that having is not so
pleasing a thing after all as wanting. It is not
logical, but is often true.
— Spock

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=QZ0LaMSN1EM
Title: STANFORD UNIVERSITY Admission Interview Secrets Revealed!
 Presenter: Super Academy
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1 The Problem

Given the relations

$$\log x + \log y = 5\,, \tag{1a}$$
$$\log x \cdot \log y = 5\,, \tag{1b}$$

find the values of x, y.

Note: This problem has already been solved, though differently, in Paper 86.

2 The Solution

In (1a), we can raise 10 to the RHS and LHS as exponents:

$$10^{\log x + \log y} = 10^{\log x} \cdot 10^{\log y} = 10^5\,, \tag{2}$$

or

$$xy = 10^5\,. \tag{3}$$

Now, if we do the same trick to (1a), we get

$$10^{\log x \log y} = (10^{\log x})^{\log y} = 10^5\,, \tag{4}$$

or

$$(x)^{\log y} = 10^5\,. \tag{5}$$

Let's now take the multiplicative inverse on both sides, the reason for which will be apparent soon.

$$(x^{-1})^{\log y} = 10^{-5}\,. \tag{6}$$

Now, from (3) we can solve for x^{-1} and substitute it in:

$$(y10^{-5})^{\log y} = 10^{-5}\,. \tag{7}$$

Next, we take the logarithm across this:

$$(\log y)[\log(y) + \log 10^{-5}] = -5\,. \tag{8}$$

This will give us the following quadratic in $\log y$:

$$(\log y)^2 - 5(\log y) + 5 = 0\,, \tag{9}$$

which has the roots,

$$\log y = \tfrac{1}{2}(5 \pm \sqrt{5})\,, \tag{10}$$

Therefore,

$$y = 10^{\frac{1}{2}(5\pm\sqrt{5})}\,, \tag{11}$$

Using (3), we can get x:

$$x = 10^{\frac{1}{2}(5\mp\sqrt{5})}\,. \tag{12}$$