Math Diversion Problem 352

P. Reany

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> Mathematics is the art of reducing any
> problem to linear algebra.
> — William Stein

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=2Uq0X0AINEg
Title: How to solve this nice math Exponential algebra problem
 Presenter: Mathematics & Statistics Guru
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1 The Problem

Given the relations

$$ab = 500\,, \tag{1a}$$
$$bc = 1000\,, \tag{1b}$$
$$ca = 1500\,, \tag{1c}$$

find the values of

$$\phi = a^2 + b^2 + c^2\,. \tag{2}$$

2 The Preparation

I will be using the following identity:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca\,. \tag{3}$$

3 The Solution

According to (3), we want to find ϕ as

$$\phi = (a+b+c)^2 - 2(ab+bc+ca) = (a+b+c)^2 - 6000\,. \tag{4}$$

Great! So what is $a+b+c$?

An intermediary result I'll need is the product abc, which I can get by multiplying the three givens together.

$$(abc)^2 = [(500)(1000)(1500)] = 10^6 \cdot 750\,. \tag{5}$$

Now, the three Givens can be rewritten as

$$a = 500/b\,, \tag{6a}$$
$$b = 1000/c\,, \tag{6b}$$
$$c = 1500/a\,, \tag{6c}$$

which we then add together:

$$a + b + c = \frac{500}{b} + \frac{1000}{c} + \frac{1500}{a} \tag{7a}$$
$$= \frac{500ac}{abc} + \frac{1000ab}{abc} + \frac{1500bc}{abc} \tag{7b}$$
$$= \frac{100}{abc}(5ac + ab + 15bc) \tag{7c}$$
$$= \frac{10^4}{abc}[\,275\,]\,. \tag{7d}$$

Hence,

$$(a + b + c)^2 = \frac{10^8}{(abc)^2}(275)^2 = \frac{10^8}{10^6 \cdot 750}(275)^2 = \frac{30,250}{3}\,. \tag{8}$$

Therefore, from (4)

$$\phi = \frac{30,250}{3} - 6000 = \frac{12,250}{3}\,. \tag{9}$$