Math Diversion Problem 353

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Physical concepts are free creations of the human mind,
and are not, however it may seem, uniquely
determined by the external world.
— Albert Einstein

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=iwj7lgRWu5U
Title:Harvard University Admission Interview Tricks
 Presenter: Super Academy
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1 The Problem

Given the relation

$$8^x = 6x\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Lemma 1: I intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

Lemma 2: I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{7}$$

for the principal value of W and $y \ln y \geq -1/e$.

3 The Solution

To use Lemma 1 above, we need to get all the functional factors on the same side. So, (1) becomes

$$\frac{1}{6} = x8^{-x}\,, \tag{8}$$

or, after swapping sides,

$$-x8^{-x} = -\frac{1}{6}\,. \tag{9}$$

Now, take the Lambert W function base 8 across this equation, to get

$$\begin{aligned} -x = W_8(-\tfrac{1}{6}) &= \frac{W(-\frac{1}{6} \cdot \ln 8)}{\ln 8} = \frac{W(-\frac{1}{6} \ln 2^3)}{\ln 2^3} \\ &= \frac{W(\frac{1}{2} \ln \frac{1}{2})}{3 \ln 2} = \frac{\ln \frac{1}{2}}{3 \ln 2} = -\frac{1}{3}\,, \end{aligned} \tag{10}$$

where we used Lemma 2.

And finally,

$$x = \frac{1}{3}\,. \tag{11}$$

Both the Presenter and WolframAlpha got another real solution of $x = 2/3$, which is easy to verify. Therefore, either something is wrong in my reasoning or in the technique itself. If it's the technique itself, this is the first time I'm aware of it.