Math Diversion Problem 355

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Everything should be made as simple as possible, but no simpler.
— Albert Einstein

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=A-gNV_adBmA
Title: A Nice Exponential Equation | Problem 442
 Presenter: aplusbi
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1 The Problem

Given the relation

$$(-1)^{z+i} = 1 + i\,, \tag{1}$$

find the values of z.

2 The Solution

Note: $1 + i = \sqrt{2}\,\dfrac{1+i}{\sqrt{2}} = \sqrt{2}\,e^{i\pi/4}$.

Note: $-1 = e^{i\pi}$.

The Given relation can be rewritten as:

$$(e^{i\pi})^{z+i} = \sqrt{2}\,e^{i\pi/4}\,, \tag{2}$$

On taking the natural logarithm, we get

$$i\pi(z+i) = \ln\sqrt{2} + \frac{i\pi}{4} + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,, \tag{3}$$

or

$$z + i = \frac{-i}{\pi}\ln\sqrt{2} + \frac{1}{4} + 2n \quad \text{where} \quad n \in \mathbb{Z}\,, \tag{4}$$

And finally,

$$z = -i\left(1 + \frac{\ln 2}{2\pi}\right) + \frac{1}{4} + 2n \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{5}$$

But wait! There's more!

WolframAlpha claims that the answer is

$$z = 2n - \frac{i\big(\pi + \ln(1+i)\big)}{\pi} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{6}$$

So, let's morph it to see if I can reach my own solution.

$$\begin{aligned} z &= 2n - \frac{i\big(\pi + \ln(1+i)\big)}{\pi} \quad \text{where} \quad n \in \mathbb{Z} && \text{(7a)} \\ &= 2n - i - \frac{i(\ln\sqrt{2} + i\pi/4)}{\pi} \quad \text{where} \quad n \in \mathbb{Z} && \text{(7b)} \\ &= 2n - i - \frac{i\ln 2}{2\pi} + \frac{1}{4} \quad \text{where} \quad n \in \mathbb{Z} && \text{(7c)} \\ &= 2n - i\Big(1 + \frac{\ln 2}{2\pi}\Big) + \frac{1}{4} \quad \text{where} \quad n \in \mathbb{Z}\,. && \text{(7d)} \end{aligned}$$