

Math Diversion Problem 356

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There's a game that I know some category theorists play,
which is when one person is supposed to state a theorem
and the other person is supposed to figure out how to
prove the theorem using the Yoneda lemma, and
it's sort of surprising how often that works.

— Emily Riehl, July 2020

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=nhsaX56UUVs>

Title: Übungsaufgabe Umkehrfunktionen

Presenter: Herr Mathe

1 The Problem

Given the relation

$$f(x) = \frac{3x+4}{x-1} \quad \text{where } x \neq 1, \quad (1)$$

find the values of $f^{-1}(2)$.

2 The Preparation

Modular Forms¹

In the construction of the modular forms, we can begin with the construction of the special linear group $\mathrm{SL}_2(\mathbb{Z})$ of 2×2 matrices over the integers with determinant equal to unity:

$$\mathrm{SL}_2(\mathbb{Z}) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}. \quad (2)$$

¹In the study of modular forms, I am still just a beginner, but the desire to utilize the little I know of it to solve this problem was enough to get me to go out on a limb for it.

Then, if τ is a point in the upper-half complex plane, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on τ , thusly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \rightarrow \frac{a\tau + b}{c\tau + d} = \tau'. \quad (3)$$

It can be shown with simple algebra that the composition of such transformations has the same form, though usually with different coefficients. So, if

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix} : \tau' \rightarrow \frac{m\tau' + n}{p\tau' + q} = \tau'' = \frac{w\tau + x}{y\tau + z}, \quad (4)$$

then what form would we evaluate coefficients w, x, y, z to be if we put τ' from (3) into (4)? Well, they'll be algebraic combinations of a, b, c, d, m, n, p , and q , right? Of course, but exactly what algebraic forms? We can use matrix algebra to calculate them:

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ma + nc & mb + nd \\ pa + qc & pb + qd \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}. \quad (5)$$

If this interests you, you might want to look up Modular Forms, which is an advanced topic in complex functions.

By the way, the mapping that τ undergoes in (3) is called a *linear fractional transformation*, and an algebraic form such as $\frac{ax + b}{cx + d}$ is called a *linear rational function of x* .

3 The Solution

To use these matrix techniques to solve this problem, I needed to loosen the restrictions placed on the 2×2 matrices. Now, instead of them having unitary determinant, they will have the lesser requirement of having nonzero determinant, which is sufficient to guarantee that these matrices all have inverses.

Warning: This method is highly nonstandard (and, so far as I know, nonrigorous as well), so do **not** use it on formal exams!

Now, we're looking for the inverse of $f(x)$, which we assume exists. Let $g(x)$ be the inverse of $f(x)$. Then,

$$g(f(x)) = x. \quad (6)$$

Since $f(x)$ in (1) is a linear rational function, can we assume that it's appropriate to use the methods of 'modular forms' presented above, even though we're not really in the realm of modular forms? I don't know. All I *do* know is that I thought it was worth a try. So I tried it! (My 'out on a limb' time.)

Now, the matrix form for the function f is

$$\begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix}. \quad (7)$$

The unknown x can be formally put into linear rational form by

$$x = \frac{x + 0}{0 + 1}, \quad (8)$$

whose matrix representation is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. On the questionable assumption that we can follow the method of (5), with $g(x)$ being represented by, say,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (9)$$

then (6) takes the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (10)$$

But if g is the inverse of f , then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(f)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \quad (11)$$

Therefore, we can write $g(x)$ as a linear rational function as

$$g(x) = \frac{dx - b}{-cx + a}, \quad (12)$$

where I ignored the factor of $1/(\det f)$ because it will cancel out of the linear rational form, numerator to denominator. On using the integer values given to us for f in (7), we have that

$$g(x) = f^{-1}(x) = \frac{-x - 4}{-x + 3} = \frac{x + 4}{x - 3}. \quad (13)$$

And lastly,

$$f^{-1}(2) = g(2) = \frac{2 + 4}{2 - 3} = -6. \quad (14)$$

Comment: If one were speed-solving this problem, one could start with (11) and end with (14). The only reason my solution looks so long is because I included a lot of explanatory details. I'm interested to see if this method will work or fail to work on my next attempt to solve a similar problem.