Math Diversion Problem 357

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> There still remains one point to be cleared up. One of the most fundamental questions has not been settled as yet: does an inertial system exist? We have learned something about the laws of nature, their invariance with respect to Lorentz transformation, and their validity for all inertial systems moving uniformly relative to each other. We have the laws but we do not know the frame to which to refer them.
>
> — Albert Einstein

1 The Problem

This problem comes from paper #5 on the webpage

https://advancedmath.org/Math/PDF/AlgebraProblems/Problem5g.pdf

which is a word problem I worked out some years ago, but it bears repeating here.

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   Steve can mow a lawn in three hours and Joe can mow
the same lawn in two hours. How long will each of them
take to mow the lawn if they both work on it together,
except that Joe works 20 minutes before Steve starts
to work?
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2 The Preparation

In the domain of algebra word problems, I define a 'mixed-rate' problem as a word problem that tells of two or more 'machines' that work together — usually simultaneously, but not always — to accomplish a single goal, task, or job. I will usually use the word 'job'.

So, you say, “What do you mean ‘machines’?! I know lots of such word problems that speak of people doing these so-called ‘jobs’, and people are **not** machines!”

Well, if you think that way, then you’ll have to come up with a separate version of this definition, which I deem to be unnecessary. If you can’t easily generalize situations through abstractions to encompass ever wider realms of the subject, then you are needlessly hindering your advancement in mathematics.

In essence, what’s the difference between two or more people painting a house at different rates, and two or more printing machines printing out a print job at different rates, and two or more fonts at different point sizes filling out a page (at different rates), and two or more water pumps filling or emptying a tank at different rates, and two or more kinds of nuts supplying protein to the final mixture at different rates, and how to invest an initial amount of money in two or more investment options, subject to legal constraints, which produce interest on the investments at different rates? I could list many more examples, I’m sure. The point is that, conceptually speaking, these are **all the same problem**, save for minor technical issues (that usually reveal themselves as mathematical constraints).

So here’s a perfect example of how to use the heuristic that ‘Every total is equal to the sum of its parts’: Starting with the given that there are two different ‘machines’, M1 and M2, working at different rates to complete a single job, what can we always conclude? We can conclude this:[1]

$$1 \text{ job} = (\text{part of job done by M1}) + (\text{part of job done by M2})\,. \tag{1}$$

If the solution presented below seems to go too fast for you, skip down to the Appendix for some definitions I use and some problem-solving aids (heuristics).

3 The Solution

The solution is not difficult if we restrain ourselves from rushing off to find the ‘key variables’. One obvious total is that Steve and Joe will work together to complete a total of one job.

$$1 \text{ job} = (\text{part of job done by Joe}) + (\text{part of job done by Steve})\,. \tag{2}$$

For our next refinement, we use the average rate R at which each mower does the job and then multiply by the time T he takes on the job.

$$1 \text{ [job]} = R_J T_J + R_S T_S\,, \tag{3}$$

[1] If the cooperating ‘machines’ all work at the same rate, then, by definition, the problem is not a ‘mixed-rate’ problem. But if one knows how to solve ‘mixed-rate’ problems, one should be able to solve single-rate problems.

where $R_J = \frac{1}{2}\frac{\text{job}}{\text{hour}}$ and $R_S = \frac{1}{3}\frac{\text{job}}{\text{hour}}$.[2] Substituting these into (3) and suppressing the units, we get

$$1 = \frac{1}{2}\,T_J + \frac{1}{3}\,T_S\,. \tag{4}$$

Now, we were given a simple relationship between T_J and T_S, namely, $T_J = T_S + \frac{1}{3}$, where the 1/3 results from converting the twenty minutes to hours. However, I want to prove that you could also arrive at this equation by searching for all the totals and finding this one:

$$\text{Joe's total time} = \begin{bmatrix}\text{part of Joe's total time}\\ \text{spent working alone}\end{bmatrix} + \begin{bmatrix}\text{part of Joe's total time}\\ \text{spent working with Steve}\end{bmatrix}$$

$$T_J = \frac{1}{3} + T_S \tag{5}$$

So, solving Eqs. (4) and (5) together, we get $T_S = 1$ hr and $T_J = (4/3)$ hr. Thus, Steve works one hour, and Joe works an hour and 20 minutes.

If you're interested in more word problems, see my many articles on solving algebra word problems at:

`https://www.advancedmath.org/Math/AlgebraWordProblems.html`

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4 Appendix: Problem-Solving Aids (Heuristics)

One way to get really confused or even lost in solving a word problem is to do too much in a single step. The fix to this is to slow down and be very intentional about everthing you write down. Develop the equations slowly. This philosophy should be apparent in the manner of the development of the heuristics and the problem solving technique employed below.

When I approach a word problem, I have a set of heuristics to guide me through the process:

1. If there are any totals or parts of a total lying around, put them into an equation (or into an inequality, if appropriate). Then, note that the following rule generates an equation: *Every total is equal to the sum of all its parts.*

$$\text{Total} = \sum_i \text{Parts}_i\,. \tag{6}$$

2. Is there some invariant Inv evidently holding from the initial state to the final state of a before-and-after process? If so, write

$$\text{Inv}_{\text{before}} = \text{Inv}_{\text{after}}\,. \tag{7}$$

[2]Why did I choose the rates to be in 'job/hour' rather than in 'hours/job'? That was to make sure that the units would be consistent between the RHS and the LHS of the equation.

And that's another equation to work with. For example, say we have a beaker containing 50 ml of salt solution (in water), to which we add 10 ml of water. Now, although the amount of water in the beaker has not remained invariant throughout this procedure, the amount of salt has. So, you can write an equation out of that invariance!

3. Is there a common or problem-specific formula to use? Such as from physics, chemistry, etc., or from mathematics, like from geometry or from number theory, such as for the summation of a series or for a weighted average of a set of numbers, or the greatest common factor or least common multiple of two or more numbers, and so on. For example, the area of a triangle $A = \frac{1}{2}bh$ is an equation. Furthermore, is there a problem-specific relationship given in the problem, such "the base of the triangle is one-third its height." That's an equation!

4. Is there a proportion given? A proportion is the stated equality of two ratios, such as

$$\frac{A}{B} = \frac{C}{D}\,. \tag{8}$$

5. Are there one or more linear or quadratic equations given? If so, write them down.

When we've collected as many equations as we have unknowns, we should be ready to solve the system simultaneously. (However, no one of these equations should be derivable from the others of the system.)

In the end, there are basically just two ways to approach solving word problems (at least at the medium level of difficulty I'm addressing). One way (the way I used to do when I was in high school) is to immediately try to intuit the 'correct' essential variables, and then, after having accomplished that, to discover what equation/s they should be fitted into. Truthfully, that method never really worked well for me.

The other way is to decide the equations (or inequalities) that are present in the problem, and then figure out what the key variables are that go into those equations. If you counter that one cannot know how to state the equation until one knows what the 'key' or 'essential variables' to the problem are, then I'll reply that that's a misconception I used to have myself. The key to this alternative method I'm promoting is to learn how to state these equations first at a level of abstration so high that those equations do not care what the 'key' variables are eventually determined to be. We saw examples of that in Eqs. (1) and (2).