Math Diversion Problem 361

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Don't ever take a fence down until you know
the reason it was put up.
— Chesterton

The YouTube video is found at:

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\begin{verbatim}
   Source: https://www.youtube.com/watch?v=3gYWZlYSlwE
   Title: Germany - Math Olympiad Exponential Problem.
    Presenter: KK Logic
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1 The Problem

Given the relation

$$2^x = x^{32}\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{5}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{6}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{7}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

3 The Solution

This paper is a redo of #222, but this time using the Lambert W function.[2] But before we do that, let's not forget to look for trivial solutions (meaning small integers), but I don't see any.

Anyway, the trick to using the lemma (6) is to convert the Given form into the appropriate form. So, in (1), we want to convert x^{32} to just x. Simple.

$$(2^{1/32})^x = x\,. \tag{8}$$

Let

$$\beta = 2^{1/32}\,, \tag{9}$$

then (8) becomes (with some algebra)

$$1 = x\beta^{-x}\,, \tag{10}$$

or

$$-x\beta^{-x} = -1\,. \tag{11}$$

And now we're ready to use the lemma:

$$-x = W_\beta(-1) = \frac{W(-1\cdot\ln\beta)}{\ln\beta}\,, \tag{12}$$

or

$$\begin{aligned} x &= -\frac{W(-1\cdot\ln 2^{1/32})}{\ln 2^{1/32}} = -32\frac{W(\frac{1}{32}\cdot\ln\frac{1}{2})}{\ln 2} = -32\frac{W(\left(\frac{1}{2}\right)^8\ln\left(\frac{1}{2}\right)^8)}{\ln 2} \\ &= -32\frac{\ln\left(\frac{1}{2}\right)^8}{\ln 2} = 32\cdot 8\,\frac{\ln 2}{\ln 2} = 256\,. \end{aligned} \tag{13}$$

[1] This notation I invented myself.

[2] Using the Lambert W function won't always be easier than its alternatives, but it is more general in that it allows for solutions other than integers or rationals.