Math Diversion Problem 361

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Dear Algebra, stop asking us to find your X,
she's not coming back.
— Woody Paige

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=4h-EAAnwpi4
Title: An Interesting Exponential Equation | No Lambert
 Presenter: SyberMath
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1 The Problem

Given the relation

$$x^x = \left(\frac{2}{3}\right)^{8/9}, \tag{1}$$

find the real values of x.

Actually, I will be using Lambert.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the following lemma:

$$W(y\ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y\ln y \geq -1/e$.

3 The Solution

First, I take the logarithm across the Given equation:

$$x \ln x = \frac{8}{9} \ln \left(\frac{2}{3} \right) . \tag{5}$$

Next, we take the Lambert W function and use the lemma above:

$$W(x \ln x) = \ln x = W(2 \, \tfrac{4}{9} \ln \tfrac{2}{3}) = W(\tfrac{4}{9} \ln (\tfrac{2}{3})^2) = W(\tfrac{4}{9} \ln (\tfrac{4}{9})) = \ln \tfrac{4}{9} . \tag{6}$$

or

$$x = \frac{4}{9} . \tag{7}$$

As for comparisons, Presenter gets two results, and I can't understand what WolframAlpha got as results. Maybe someday.