Math Diversion Problem 364

P. Reany

January 22, 2025

> It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.
> — Emil Artin

> More than a century since its debut, representation theory has served as a key ingredient in many of the most important discoveries in mathematics. Yet its usefulness is still hard to perceive at first.
> — Kevin Hartnett

(Representation theory — among other uses, it is the representation of elements of an arbitrary group by the elements of a linear map to a vector space. Once a basis is chosen, the linear map can take the form of a matrix group.)

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=gmGLBhS4fXY
Title: A Nice Nonstandard Equation
 Presenter: SyberMath
```

1 The Problem

Given the relation

$$4^x = -x\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

I'll need the following lemma:

$$W(y\ln y) = \ln y\,, \tag{7}$$

for the principal value of W and $y\ln y \geq -1/e$.

3 The Solution

To use the Lambert lemma above, we need to massage the Given relation into proper form.

$$1 = -x4^{-x}\,. \tag{8}$$

On taking the Lambert W function base 4 and reversing sides, we have that

$$-x = W_4(1) = \frac{W(1\cdot\ln 4)}{\ln 4} = \frac{W(2\ln 2)}{2\ln 2} = \frac{\ln 2}{2\ln 2} = \frac{1}{2}\,. \tag{9}$$

Hence,

$$x = -\frac{1}{2}\,. \tag{10}$$