Math Diversion Problem 368

P. Reany

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Young men should prove theorems, old
men should write books.
— G.H. Hardy

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=SRPrcW8ESLM
Title: An Exponentially Interesting And Nice Equation
 Presenter: SyberMath
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1 The Problem

Given the relation

$$x^{x^x} = 2^{-\sqrt{2}}\,, \tag{1}$$

find the values of x.

2 The Solution

In problems like these, I like to make a standard variable change, such as:[1]

$$x = 2^\alpha\,. \tag{2}$$

Then (1) becomes,

$$(2^\alpha)^{(2^\alpha)^{(2^\alpha)}} = 2^{-\sqrt{2}}\,, \tag{3}$$

or,

$$2^{\alpha\, 2^{\alpha 2^\alpha}} = 2^{-\sqrt{2}}\,. \tag{4}$$

On equating the exponents, we get

$$\alpha\, 2^{\alpha 2^\alpha} = -\sqrt{2}\,. \tag{5}$$

At this point, I thought that getting rid of the negative sign of some value, so I set

$$\beta \equiv -\alpha\,, \tag{6}$$

[1] This is the hardest such problem of this kind I've so far encountered.

then (5) becomes

$$\beta\, 2^{-\beta 2^{-\beta}} = \sqrt{2}\,. \tag{7}$$

Next, we take the logarithm base 2:

$$\log_2 \beta + (-\beta\, 2^{-\beta}) = \tfrac{1}{2}\,. \tag{8}$$

Now, some reasonable rational numbers to try for β would be $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

Let's begin with $\beta = 2$:

$$\log_2 2 + (-2 \cdot 2^{-2}) = 1 - 2\frac{1}{4} = 1 - \frac{1}{2} = \frac{1}{2}\,, \tag{9}$$

which works. Thus,

$$\alpha = -\beta = -2\,. \tag{10}$$

And thus,

$$x = 2^{-2} = \frac{1}{4}\,. \tag{11}$$