Math Diversion Problem 371

P. Reany

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> When I give this talk to a physics audience, I
> remove the quotes from my 'Theorem'.
> — Brian Greene

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=BQ0B9JL-VjQ
Title: How to solve? | Oxford entrance exam question
 Presenter: Math Beast
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1 The Problem

Given the relation

$$x^x = e^{-\pi + i \ln 4} , \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B , \tag{2}$$

$$z = W(B) , \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the following lemma:

$$W(y \ln y) = \ln y , \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

3 The Solution

To my mind, this is clearly a job for the Lambert W function. We can begin by taking the natural logarithm across (1):

$$x \ln x = -\pi + i \ln 4 + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z} \,. \tag{5}$$

Then we can apply the Lambert W function across this equation and use the lemma above, to get

$$\ln x = W(-\pi + i \ln 4 + 2\pi i n) \quad \text{where} \quad n \in \mathbb{Z} \,. \tag{6}$$

Lastly, we just need to raise e to the power of Eq. (6),[1] to get

$$x = e^{W(-\pi + i \ln 4 + 2\pi i n)} \quad \text{where} \quad n \in \mathbb{Z} \,. \tag{7}$$

[1] Yes, it sounds strange, but it works for me. The expression 'to take an object to the power of an equation' can be explained this way: Let our equation be 'LHS = RHS', and let our object be z. Then "z raised to the power 'LHS = RHS'" means the following:

$$z^{\text{LHS}} = z^{\text{RHS}} \,.$$

This is often particularly useful when dealing with logarithms.