Math Diversion Problem 373

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It was an act of desperation. For six years I had struggled with the blackbody theory. I knew the problem was fundamental and I knew the answer. I had to find a theoretical explanation at any cost, except for the inviolability of the two laws of thermodynamics.
— Max Planck

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=g8KWDPCXzCg
Title: An Interesting Exponential Equation
 Presenter: SyberMath
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1 The Problem

Given the relation

$$2x^{2x} = 1\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

3 The Solution

We begin by dividing through by 2 and then taking the squareroot across the equation:

$$x^x = \pm\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}^{1/2}\,, \tag{5}$$

where we threw out the negative root. Next, we take the natural logarithm across this equation.

$$x \ln x = \ln\,(\tfrac{1}{2})^{1/2} = \tfrac{1}{2}\ln\tfrac{1}{2}\,. \tag{6}$$

On taking the Lambert W function across this equation and using the lemma above, we get

$$\ln x = \ln\tfrac{1}{2}\,. \tag{7}$$

From this we conclude that

$$x = \tfrac{1}{2}\,. \tag{8}$$

WolframAlpha also gives as a real solution

$$x = e^{W_{-1}(-\ln\,(2)/2)}\,, \tag{9}$$

which (seems) to come from that negative root we threw out in (5).