Math Diversion Problem 374

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Theory like mist on eyeglasses — obscures facts.
— Charlie Chan

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=7KNr2AgmuZI
Title: An Interesting Equation With Euler's Number
 Presenter: SyberMath
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1 The Problem

Given the relation

$$x^{\ln x^{\ln x}} = e\,, \tag{1}$$

find the real values of x.

2 The Solution

First solution:

We begin by taking the natural logarithm across (1).

$$\ln x^{\ln x} \ln x = (\ln x)(\ln x)(\ln x) = 1\,, \tag{2}$$

or

$$(\ln x)^3 = 1\,. \tag{3}$$

On setting unity to $e^{2\pi in}$ for integer n, we get

$$\ln x = e^{2\pi in/3} \quad \text{for} \quad n \in \mathbb{Z}\,, \tag{4}$$

but we only need three unique roots to (3), namely

$$\ln x = e^{2\pi ik/3} \quad \text{for} \quad k \in \{0, 1, 2\}\,, \tag{5}$$

Now we raise e to this last equation:

$$x = e^{e^{2\pi ik/3}} \quad \text{for} \quad k \in \{0, 1, 2\}\,. \tag{6}$$

For $k = 0$, we get the only real root.

$$x = e^{e^0} = e^1 = e\,. \tag{7}$$

Second solution:

We begin by setting $x = e^{\alpha}$, getting

$$e^{\alpha \ln(e^{\alpha \ln(e^{\alpha})})} = e^1\,, \tag{8}$$

which telescopes down to

$$e^{\alpha^3} = e^1\,, \tag{9}$$

or

$$\alpha^3 = 1\,. \tag{10}$$

So we've reached a similar cubic as before. As before, we'll have to take the real root, and get

$$\alpha = 1\,. \tag{11}$$

Hence,

$$x = e\,. \tag{12}$$