Math Diversion Problem 381

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February 4, 2025

And a little humility would go a long way.
— Hardy to Ramanujan
(According to the movie,
*The Man Who
Knew Infinity*)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=pUhA_ETbWj8
Title: Lambert W Function - Introduction
 Presenter:  Owls Math
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Presenter gives a nice intro to the Lambert W function.

1 The Problem

Given the relation

$$x^x = 5\,, \tag{1}$$

find the values of x.

[Skip down to the solution, if you prefer.]

2 The Preparation

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{2}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{3}$$

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3 The Solution

Let's begin by taking the natural logarithm across the Given relation.

$$x \ln x = \ln 5 + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{4}$$

Next, we take the Lambert W function across this (and use the lemma).

$$\ln x = W(\ln 5 + 2\pi i n) \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{5}$$

And finally, we just raise e to the last equation,[1] to get

$$x = e^{W(\ln 5 + 2\pi i n)} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{6}$$

Out of all this, the real solution is

$$x = e^{W(\ln 5)}\,. \tag{7}$$

[1] To raise a number b to the 'power of an equation' simply means this: If the equation is 'LHS = RHS', then $b^{\text{LHS=RHS}}$ means $b^{\text{LHS}} = b^{\text{RHS}}$.