Math Diversion Problem 383

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After a time, you may find that having is not so
pleasing a thing after all as wanting. It is not
logical, but is often true.
— Spock

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=HOvHH4Jt1Pk
Title: Stanford University Exponential Problem.
 Presenter: Super Academy
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1 The Problem

Given the relations

$$3^x - 3^y = 16\,, \tag{1a}$$

$$3^{x+y} = 4\,, \tag{1b}$$

find the values for x, y.

2 The Solution

Let's start off by framing our solutions in the complex numbers. That way, I can do this:

$$a \equiv 3^{x/2} + i3^{y/2}\,. \tag{2}$$

On squaring a, we get

$$a^2 = 3^x - 3^y + 2i(\sqrt{3})^{x+y} \tag{3a}$$

$$= 16 \pm 4i\,, \tag{3b}$$

where we use (1a) and the square root of (1b).

On taking the square root of this last equation, we have that[1]

$$a = \pm 2\sqrt{4 \pm i}\,. \tag{4}$$

Thus, from (2), we get for the plus sign in the radicand above:

$$3^{x/2} = \mathrm{Re}\,(a) = \pm 2\mathrm{Re}\,(\sqrt{4+i}) = \pm 2\sqrt{\tfrac{1}{2}(4+\sqrt{17})}\,. \tag{5}$$

Squaring this, we have that

$$3^x = 2(4+\sqrt{17})\,. \tag{6}$$

Solving for x, we get

$$x = \frac{\log\,[2(4+\sqrt{17})]}{\log 3}\,. \tag{7}$$

Similarly, from (2), we get for the minus sign in the radicand:

$$x = \frac{\log\,[2(4+\sqrt{17})]}{\log 3}\,, \tag{8}$$

so, no difference. Therefore, we can expect to get exactly one y value as well.

Solving (1b) for y, we get

$$y = \frac{\log 4}{\log 3} - x = \frac{\log 4}{\log 3} - \frac{\log\,[2(4+\sqrt{17})]}{\log 3} = \frac{\log 4}{\log 3} - \frac{\log 2 + \log\,(4+\sqrt{17})}{\log 3}\,, \tag{9a}$$

or

$$y = \frac{\log 2 - \log\,(4+\sqrt{17})}{\log 3} = \frac{\log 2 + \log\,(-4+\sqrt{17})}{\log 3}\,. \tag{9b}$$

[1]Note: I usually take the square roots of complex numbers by using a program in Mathematica.