Math Diversion Problem 384

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> The human mind has never invented a labor-saving
> machine equal to algebra.
> — J. Willard Gibbs

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=ca_BOHCmRbo
Title: An Imaginarily Exponential Equation
     | Problem 386
 Presenter: aplusbi
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1 The Problem

Given the relations

$$z = i^z \,, \tag{1}$$

find the complex values of z.

(Skip down to the solution, if you like.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A \,, \tag{4}$$

[1] This notation I invented myself.

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{5}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{6}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

3 The Solution

In the lemma above, I stipulated that the base should be a real number greater than zero. This is a reasonable assumption, but I intend to be playful right now and let the base be i. Let's see what happens and let the Given relation be rewritten as

$$-zi^{-z} = -1\,, \tag{7}$$

in anticipation of introducing the Lambert W function. Therefore,

$$-z = W_i(-1) = \frac{W(-1 \cdot \ln i)}{\ln i}\,. \tag{8}$$

But $i = e^{i\pi/2}$, so

$$-z = \frac{W(-i\pi/2)}{i\pi/2}\,. \tag{9}$$

And finally,

$$z = \frac{2i}{\pi} W(-i\pi/2)\,. \tag{10}$$

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4 Appendix: Basics of Complex Numbers

Typically, we find a generic complex number denoted by the letter z, but one is free to choose other letters, as well. So, if z is a complex number, in general it has both real and imaginary parts:

$$z = a + bi\,, \tag{11}$$

where a, b are real components of basis vectors $1, i$. But they are also expressed as, respectively, the 'real' and 'imaginary' components of z.

Complex conjugation of complex number z is an operation that leaves real numbers alone but replaces the unit imaginary i with its negative, i.e., $-i$. The symbols most often used to represent complex conjugation are the $*$ and the overbar. I'll usually use the overbar. Thus, the complex conjugate of z in (11) is

$$\overline{z} = a - bi\,. \tag{12}$$

Obviously, the complex conjugation of a pure real number has no effect.

A funny thing happens when we multiply a complex number by its conjugate:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2\,. \tag{13}$$

So, $z\overline{z}$ is zero if and only if $z = 0$, otherwise, it's a positive real number.

Another funny thing happens when we add a complex number and its conjugate: we also get a real number. Let's see.

$$z + \overline{z} = (a + bi) + (a - bi) = 2a\,. \tag{14}$$

Why do we care about this? Because sometimes we need to map complex numbers into the real numbers to get information on the complex numbers. This problem will show you that.

I'm not going to prove this here, but every complex number can be expressed in exponential (or polar) form:

$$z = a + bi = \sqrt{a^2 + b^2}e^{i\theta} = (z\overline{z})^{1/2}e^{i\theta} = r\,e^{i\theta}\,, \tag{15}$$

where we can think of r as the length of the complex numbers z or $\overline{z}$.

$$r \equiv (z\overline{z})^{1/2} \quad \text{or} \quad r^2 = z\overline{z} = |\,z\,|^2\,. \tag{16}$$

So, it will be good to know all this stuff in this section before you attempt to follow my solutions to these complex variables problems.

By the way, the complex numbers are what's called a *field*, so they can be added, subtracted, multiplied, and divided by each other (except you can't divide by zero, as usual). And, therefore, you can apply the quadratic formula to them! (Yay!)

Lemma 1: If a complex number z is equal to its own conjugate $z = \overline{z}$, it's real.

Lemma 2: If a complex number z is complex conjugated twice then there's no change: $\overline{\overline{z}} = z$.

Lemma 3: The complex conjugated of a product or a sum is the product or sum of the complex conjugates: $\overline{z_1 z_2} = \overline{z}_1\overline{z}_2$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$.

Lemma 4: If $s, t \in \mathbb{R}$ and $z = s + ti$ then

$$i\overline{z} = t + si\,. \tag{17}$$