Math Diversion Problem 386

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I feel that a visual representation of the Dirac algebra is of great benefit, because it can provide an additional insight that is not easily expressed with words or equations.
— David M. Goodmanson
['A graphical representation of the Dirac algebra', American J. Phys., Vol. 64, No. 7, July 1996, p. 870.]

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=wmAYuYDFjE0
Title: You Should Learn This Trick
 Presenter: BriTheMathGuy
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1 The Problem

Given the relation

$$x^{x^3} = 36\,, \tag{1}$$

find the real values of x.

(Skip down to the solution, if you like.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Lemma 1: I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

3 The Solution

Let's begin by cubing both sides of the Given relation:

$$(x^3)^{x^3} = 36^3\,. \tag{5}$$

Next, we take the natural logarithm.

$$(x^3)\ln(x^3) = 3\ln 36\,. \tag{6}$$

Now we take the Lambert W function across this last equation, to get

$$\ln(x^3) = W(3\ln 36) = W(6\ln 6) = \ln 6\,. \tag{7}$$

Next, we raise e to this equation, to get[1]

$$x^3 = 6\,, \tag{8}$$

of which the real root is

$$x = 6^{1/3}\,. \tag{9}$$

[1] To raise a number b to the 'power of an equation' simply means this: If the equation is 'LHS = RHS', then $b^{\mathrm{LHS=RHS}}$ means $b^{\mathrm{LHS}} = b^{\mathrm{RHS}}$.