Math Diversion Problem 387

P. Reany

February 5, 2025

With me, everything turns into mathematics.
— Rene Descartes
(P.S. I calculate; therefore I am.)

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=9R0qz6_IKgw
Title: Harvard University Admission Interview Tricks
 Presenter: Super Academy
```

1 The Problem

Given the relation

$$3^{x-2} = x \,, \tag{1}$$

find the real values of x.

(Skip down to the solution, if you like.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A \,, \tag{4}$$

[1] This notation I invented myself.

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s} \,. \tag{5}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A) \,, \tag{6}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

---

3 The Solution

Let's begin by rewriting the Given relation to the following form:

$$3^{-2} = \frac{1}{9} = x3^{-x} \,, \tag{7}$$

which can be rewritten into

$$-x3^{-x} = -\frac{1}{9} \,. \tag{8}$$

Now we take the Lambert W function base 3 across this last equation, to get

$$-x = W_3\big(-\tfrac{1}{9}\big) = \frac{W(-\frac{1}{9}\ln 3)}{\ln 3} \,, \tag{9}$$

or

$$x = -\frac{W(-\frac{1}{9}\ln 3)}{\ln 3} \,. \tag{10}$$

Now, we look at the details.

$$-\frac{1}{9}\ln 3 \approx -0.122068 \,, \tag{11}$$

but

$$-\frac{1}{e} \approx -.367894 \,, \tag{12}$$

Therefore,

$$-\frac{1}{e} \leq -\frac{1}{9}\ln 3 < 0 \,, \tag{13}$$

which puts $W(-\frac{1}{9}\ln 3)$ in the state of having two possible real values,

$$x = \begin{cases} -\dfrac{W_{-1}(-0.122068)}{\ln 3} & \approx 3.00000 \,, \\ -\dfrac{W_0(-0.122068)}{\ln 3} & \approx 0.127869 \,, \end{cases} \tag{14}$$

whose values I let WoframAlpha calculate. By the way, the integer 3 is clearly a solution to (1)