Math Diversion Problem 388

P. Reany

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> It is a capital mistake to theorize
> in advance of the facts.
> — Sherlock Holmes (Jeremy Brett)
> [Episode *The Second Stain*]

The YouTube video is found at:

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Source: https://www.youtube.com/shorts/ddDY_a_aMaE
Title: Can you solve this equation with the Lambert W?
 Presenter: Gresty Math Short
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1 The Problem

Given the relation

$$\ln x^{\ln x} = 3\,, \tag{1}$$

find the values of x.

(Skip down to the solution, if you like.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.
Lemma 1: I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

3 The Solution

Let's begin by taking the logarithm of both sides:

$$\ln x \, \ln (\ln x) = \ln 3 \, . \tag{5}$$

Now we take the Lambert W function across this equation, to get

$$\ln (\ln x) = W(\ln 3) \, . \tag{6}$$

Next, we raise e to this equation, to get[1]

$$\ln x = e^{W(\ln 3)} \, , \tag{7}$$

and then repeat that operation:

$$x = e^{e^{W(\ln 3)}} \, . \tag{8}$$

[1] To raise a number b to the 'power of an equation' simply means this: If the equation is 'LHS = RHS', then $b^{\text{LHS=RHS}}$ means $b^{\text{LHS}} = b^{\text{RHS}}$.