Math Diversion Problem 389

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Small moves, Ellie. Small moves.
— from the movie, *Contact*

The YouTube video is found at:

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Source: https://www.youtube.com/shorts/J_YpAnvuDpo
Title: This is a tough partial differentiation
 Presenter: Gresty Math Short
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1 The Problem

Given the relation

$$x^3 + y^3 + z^3 + x^2y^2z^2 = 0\,, \tag{1}$$

find the partial derivative $\partial y/\partial z$.

2 The Solution

Note: For a deeper explanation of my methodology on differentiation, see my many articles on Structured Differentiation (SD).

It's so important to approach this kind of problem logically. Say we're presented an expression in three unknowns and that's all we know about the expression. Then, so far as we can tell, the expression contains three independent variables, which means three degrees of freedom.

But the expression given to us on the LHS of (1) is constrained by the equality it resides in; hence, the system of three variables has only two degrees freedom, meaning, just two independent variables, which we have to choose. Therefore, we should set one of these variables as functionally dependent on the other two. The choice for that dependent variable is obvious: It has to be y or we can't solve the problem as given to us. That leaves x, z as the independent variables, and that the total derivative of one of them by the other is zero, of course. So, we have decided that, functionally speaking,

$$y = f(x, z)\,, \tag{2}$$

and that

$$\frac{\delta x}{\delta z} = \frac{\delta z}{\delta x} = 0\,. \tag{3}$$

Of course,

$$\frac{\delta}{\delta z} 0 = 0\,. \tag{4}$$

Now, we're told to solve for $\partial y/\partial z$. We will do this by so-called 'implicit differentiation' of the Given relation. You have to understand why this is called 'implicit differentiation'. There is such a thing as an 'implicit differentiation', but we are not going to employ such a derivative on this first attaempt! What we are going to do is to take the total derivative by z across (1). In my own opinion, it's really a bad nomenclature that convention uses in this case.

Conventional wisdom is to call it 'implicit differentiation' simply because it's not 'explicit differentiation', which we could do if we could solve for y as a function of x, z explicitly, as in the result

$$y = \phi(x, z)\,, \tag{5}$$

where ϕ is known explicitly. Take just a few seconds to convince yourself that that would be difficult, if not impossible, to do that for the Given relation. But even if it were doable, we still don't have to go to that effort. 'Implicit differentiation' can be simpler, so long as you don't get confused.

With respect to Eq. (1), are we going to differentiate the LHS, the RHS, or both? Both of course. One of the fundamental rules of SD is that if one takes the total derivative across a true equality, then one gets back a true equality. Thus, if we take the total derivative by z across (1), we get, at first

$$\frac{\delta}{\delta z}\left(x^3 + y^3 + z^3 + x^2y^2z^2\right) = 0\,. \tag{6}$$

First Attempt at a solution:

Now it's time to apply the derivative.

$$3y^2\frac{\partial y}{\partial z} + 3z^2 + 2x^2y\frac{\partial y}{\partial z}z^2 + 2x^2y^2z = 0\,. \tag{7}$$

On factoring out the partial derivative, we get

$$\frac{\partial y}{\partial z}(3y^2 + 2x^2yz^2) + 3z^2 + 2x^2y^2z = 0\,. \tag{8}$$

Finally, we have (with a little algebra) that

$$\frac{\partial y}{\partial z} = -\frac{3z^2 + 2x^2y^2z}{3y^2 + 2x^2yz^2} = -\frac{z(3z + 2x^2y^2)}{y(3y + 2x^2z^2)}\,. \tag{9}$$

Second Attempt at a solution:

Following the lead of the Presenter, I will set the LHS of (6) to w, and thus make the 'new' equation:

$$\frac{\delta}{\delta z} w = 0 \,. \tag{10}$$

Now, the delta (total) derivative can be expanded to

$$\frac{\delta}{\delta z} = \frac{\partial}{\partial z} + \frac{\not\partial}{\not\partial z} \,, \tag{11}$$

where $\partial/\partial z$ is the usual explicit derivative and $\not\partial/\not\partial z$ is the implicit derivative, which uses the chain rule to ferret out functional dependence 'under the hood', so to speak. Then,

$$\frac{\delta w}{\delta z} = \frac{\partial w}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} = 0 \,. \tag{12}$$

We can solve this for $\dfrac{\partial y}{\partial z}$, geting

$$\frac{\partial y}{\partial z} = -\frac{\dfrac{\partial w}{\partial z}}{\dfrac{\partial w}{\partial y}} \,. \tag{13}$$

Now, with

$$\frac{\partial w}{\partial z} = 3z^2 + 2x^2y^2z \,, \tag{14}$$

$$\frac{\partial w}{\partial y} = 3y^2 + 2x^2yz^2 \,, \tag{15}$$

we can determine $\dfrac{\partial y}{\partial z}$ from (13):

$$\frac{\partial y}{\partial z} = -\frac{3z^2 + 2x^2y^2z}{3y^2 + 2x^2yz^2} \,, \tag{16}$$

and finish off as before.