Math Diversion Problem 390

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Mathematics compares the most diverse phenomena and
discovers the secret analogies that unite them.
— Joseph Fourier

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=djYE4af8oQQ
Title: Lambert W Time!
 Presenter: Owls Math
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1 The Problem

Given the relation

$$\left(\frac{1}{256}\right)^x + x = \frac{5}{8}\,, \tag{1}$$

find the values of x.

(Skip down to the solution, if you like.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

Lemma 1: I'll need the following lemma:

$$W(y\ln y) = \ln y\,, \tag{7}$$

for the principal value of W and $y\ln y \geq -1/e$.

3 The Solution

Note: $256 = 2^8 = 2^{2^3}$.

Let's begin by subtracting x from both sides and switch sides

$$x - \frac{5}{8} = -\left(\frac{1}{256}\right)^x = -256^{-x}\,. \tag{8}$$

Next, we set $y = x - \frac{5}{8}$ and do some algebra:

$$y = -256^{-y-5/8} = -256^{-y}\,256^{-5/8}\,. \tag{9}$$

Then

$$y256^y = -256^{-5/8} = -(2^8)^{-5/8} = -2^{-5}\,. \tag{10}$$

Now we take the Lambert W function base 256 across this equation, to get

$$y = W_{256}(-2^{-5}) = \frac{W(-2^{-5}\cdot\ln 256)}{\ln 256} = \frac{W(-2^{-5}\cdot\ln 2^{2^3})}{\ln 256} = \frac{W(-\frac{1}{4}\ln 2)}{\ln 256}\,. \tag{11}$$

Going back to x:

$$x = \frac{5}{8} + \frac{W(-\frac{1}{4}\ln 2)}{\ln 256}\,. \tag{12}$$

Now, since $-\frac{1}{e} \leq -\frac{1}{4}\ln 2 < 0$, then we get two real values for x, namely

$$x = \begin{cases} \frac{5}{8} + \frac{W_0(-\frac{1}{4}\ln 2)}{\ln 256}\,, \\ \frac{5}{8} + \frac{W_{-1}(-\frac{1}{4}\ln 2)}{\ln 256}\,. \end{cases} \tag{13}$$