# Math Diversion Problem 391

P. Reany

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Easy to criticize, more difficult to be correct.
— Charlie Chan

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=l3J_EP6W8gs
Title: UNSW Integration Bee 2023 Semi-Finals #2-3
 Presenter: Owls Math
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## 1 The Problem

Solve the integral

$$I(x) = \int \ln\left(\sqrt{x} + \sqrt{x+1}\right) dx\,. \tag{1}$$

## 2 The Solution

A quick look at the Owls Math YouTube site is enough to show that the Presenter is far more expert at integration than I am. Just the same, I wanted to perform the integral from a different viewpoint.

It will be useful to note now that the domain of the indefinite integral in (1) as the nonnegative reals.

So we'll try the change of variable:

$$x = \sinh^2 y\,. \tag{2}$$

This, then, gives us

$$\sqrt{x} = \sinh y\,, \tag{3a}$$

$$\sqrt{x+1} = \sqrt{\sinh^2 y + 1} = \cosh y\,, \tag{3b}$$

$$\sqrt{x} + \sqrt{x+1} = \sinh y + \cosh y = e^y\,, \tag{3c}$$

$$dx = 2 \sinh y \cosh y \, dy\,. \tag{3d}$$

On substituting these results into (1), we have that

$$I(y) = 2\int y \sinh y \cosh y \, dy\,. \tag{4}$$

At this point, most people might try integration by parts, but I prefer something related. Consider

$$D_y(y\cosh^2 y) = \cosh^2 y + 2y\cosh y\,\sinh y\,, \tag{5}$$

where the second term on the right is our integrand. Therefore, after integrating and reorganizing, we have that

$$I(y) = y\cosh^2 y - \int \cosh^2 y\, dy + c\,. \tag{6}$$

Now,

$$\int \cosh^2 y\, dy = \frac{1}{2}(y + \sinh y \cosh y)\,. \tag{7}$$

Therefore, (6) becomes

$$I(y) = y\cosh^2 y - \frac{1}{2}(y + \sinh y \cosh y) + c\,. \tag{8}$$

All we have to do now is to revert to variable $x$:

$$I(x) = (1+x)\sinh^{-1}\sqrt{x} - \frac{1}{2}(\sinh^{-1}\sqrt{x} + \sqrt{x}\sqrt{x+1}\,) + c\,. \tag{9}$$

However, we have an identity we can use:

$$\sinh^{-1} z = \ln\left[\, z + \sqrt{z^2+1}\,\right]. \tag{10}$$

Thus, (12) becomes

$$I(x) = (1+x)\ln\left[\,\sqrt{x} + \sqrt{x+1}\,\right] - \frac{1}{2}(\ln\left[\,\sqrt{x} + \sqrt{x+1}\,\right] + \sqrt{x}\sqrt{x+1}\,) + c \tag{11}$$

$$= (\frac{1}{2} + x)\ln\left[\,\sqrt{x} + \sqrt{x+1}\,\right] - \frac{1}{2}(\sqrt{x}\sqrt{x+1}\,) + c\,. \tag{12}$$