Math Diversion Problem 392

P. Reany

February 8, 2025

> The only way to learn mathematics is
> to do mathematics.
> — Paul Halmos

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=Owzej81Y5EA
Title: The DISTURBING TRUTH about Lambert W
 Presenter: Owls Math
```

1 The Problem

Given the relation

$$1 + x = x \ln \frac{1}{x}\,, \tag{1}$$

find the values of x.

(Skip down to the solution, if you like.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Lemma 1: I'll need the following lemma:

$$W(y \ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

3 The Solution

Let's begin by rewriting (1) into the form

$$1 + x = -x \ln x \,. \tag{5}$$

Next, let's raise e to the power of Eq. (5).[1]

$$e^{1+x} = e^{-x \ln x} = \left(e^{\ln x}\right)^{-x} = x^{-x} \,, \tag{6}$$

which can be rewritten as

$$e = x^{-x} e^{-x} = (xe)^{-x} \,. \tag{7}$$

This can be further taken to

$$(xe)^{x} = e^{-1} \,. \tag{8}$$

Now raise both sides to the e power:

$$(xe)^{xe} = e^{-e} \,. \tag{9}$$

Next, take the logarithm:

$$(xe) \ln(xe) = -e \,. \tag{10}$$

Then the Lambert W function:[2]

$$\ln (xe) = W(-e) \,. \tag{11}$$

Then raise e to this equation:

$$xe = e^{W(-e)} \,. \tag{12}$$

And finally,

$$x = e^{W(-e)-1} \,. \tag{13}$$

WolframAlpha claims that this is two solutions, namely

$$x = \begin{cases} e^{W_0(-e)-1} \,, \\ e^{W_{-1}(-e)-1} \,. \end{cases} \tag{14}$$

I presume that these are nonreal solutions, first because WolframAlpha did not claim that they are real, and second because the domain for real solutions to $W(x)$ is for $-\frac{1}{e} \leq x < 0$, which makes the presented solutions out of bounds.

[1] To raise a number b to the 'power of an equation' simply means this: If the equation is 'LHS = RHS', then $b^{\text{LHS=RHS}}$ means $b^{\text{LHS}} = b^{\text{RHS}}$.

[2] There are many ways to get from (5) to this equation.