Math Diversion Problem 393

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Dear Algebra, stop asking us to find your X,
she's not coming back.
— Woody Paige

The YouTube video is found at:

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Source: The last article
Title: Follow-up.
 Presenter: Patrick
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1 The Problem

This is a take-off from the last equation to solve, namely,

$$1 + x = x \ln \frac{1}{x} \,. \tag{1}$$

This time I want to try a similar equation to solve and apply a different method.

Given the relation

$$1 + x = x \ln x \,, \tag{2}$$

find the values of x.

(Skip down to the solution, if you like.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{3}$$

then

$$z = W(B) \,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

3 The Solution

Let's begin by making the variable substitution:

$$x = e^y \,. \tag{5}$$

Then (2) becomes

$$1 + e^y = ye^y \,. \tag{6}$$

Multiplying through by e^{-y} gives

$$e^{-y} + 1 = y \,, \tag{7}$$

or

$$e^{-y} = y - 1 \,. \tag{8}$$

Let's make another variable substitution:

$$z = 1 - y \,. \tag{9}$$

Then (8) becomes

$$e^{z-1} = -z \,. \tag{10}$$

This can be rewritten as

$$-ze^{-z} = e^{-1} \,. \tag{11}$$

On taking the Lambert W function across this equation, we have that

$$-z = W(\frac{1}{e}) \,. \tag{12}$$

Reverting back to y, we get

$$y = W\big(\frac{1}{e}\big) + 1 \,. \tag{13}$$

And finally, back to x:

$$x = e^{W(\frac{1}{e})+1} \,. \tag{14}$$