Math Diversion Problem 394

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Linear Algebra is about the relation between
the columns and the rows.
— Gilbert Strang

The YouTube video is found at:

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Source: ?
Title: ?
 Presenter: ?
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1 The Problem

Given the relation

$$W(x+1) = (x+1)^2 \,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B \,, \tag{4}$$

then

$$z = W_a(B) \,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

The trivial solution is $x = -1$, but are there any other solutions?

If we're going to extricate $x + 1$ from the argument of the Lambert W function, we're going to need to replace it by something that is extricable, like ye^y or $y \ln y$, of which the former is preferred. So, let's begin by making the variable substitution:

$$x + 1 = ye^y\,. \tag{7}$$

Then (1) becomes

$$W(ye^y) = y = y^2 e^{2y}\,. \tag{8}$$

On simplifying, we get

$$ye^{2y} = 1\,, \tag{9}$$

or

$$y(e^2)^y = 1\,. \tag{10}$$

Now we just apply the Lambert W function base e^2, to get

$$y = W_{e^2}(1) = \frac{W(1 \cdot \ln e^2)}{\ln e^2} = \frac{W(2)}{2}\,. \tag{11}$$

Therefore,

$$x = \tfrac{1}{2} W(2)\, e^{W(2)/2} - 1\,. \tag{12}$$

Introduce

$$\lambda = W(2)\, e^{W(2)/2}\,. \tag{13}$$

Next, we're going to square λ, use an identity, and then take its square root.

$$\lambda^2 = W(2)[W(2)\, e^{W(2)}] = 2W(2)\,. \tag{14}$$

Now we take its square root:

$$\lambda = \sqrt{2W(2)}\,. \tag{15}$$

Returning to (12), we have that

$$x = \tfrac{1}{2}\sqrt{2W(2)} - 1\,. \tag{16}$$