Math Diversion Problem 396

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Talent is cheaper than table salt. What separates the
talented individual from the successful one is
a lot of hard work.
— Stephen King

The YouTube video is found at:

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Source: https://www.youtube.com/shorts/chH5AIvuKyQ
Title: Can you use the Lambert W function to solve this?
 Presenter: Gresty Math Short
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1 The Problem

Given the relation

$$x + 3\ln x = 12\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

Let's first rewrite (1) as

$$x + \ln x^3 = 12\,, \tag{7}$$

and then raise e to the power of this equation, yielding[1]

$$x^3 e^x = e^{12}\,. \tag{8}$$

On taking the cube root across this equation, we get

$$x(e^{1/3})^x = e^4\,. \tag{9}$$

Next we just apply the Lambert W function base $e^{1/3}$, to get

$$x = W_{e^{1/3}}(e^4) = \frac{W(e^4 \cdot \ln e^{1/3})}{\ln e^{1/3}} = 3W(e^4/3)\,. \tag{10}$$

[1]To raise a number b to the 'power of an equation' simply means this: If the equation is 'LHS = RHS', then $b^{\text{LHS=RHS}}$ means $b^{\text{LHS}} = b^{\text{RHS}}$.