Math Diversion Problem 397

P. Reany

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That's the problem with false proofs of true theorems;
it's not easy to produce a counterexample.
— Jeffrey Shallit

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=ZGMkPLfna5k
Title: This Video Will Make You Better At Solving
 Presenter:BriTheMathGuy
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1 The Problem

Given the relation

$$x^2 = \left(\frac{1}{2}\right)^x , \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

Let's begin by taking the square root of (1)

$$x = \pm \left(\sqrt{1/2}\right)^{x}\,. \tag{7}$$

Now, we convert into the appropriate form:

$$-x \left(\sqrt{1/2}\right)^{-x} = \pm 1\,, \tag{8}$$

Then we take the Lambert W function base $\sqrt{1/2}$:

$$-x = W_{\sqrt{1/2}}(\pm 1) = \frac{W(\pm 1 \cdot \ln \sqrt{1/2})}{\ln \sqrt{1/2}}\,. \tag{9}$$

Thus,

$$x = \begin{cases} -\dfrac{W(+1 \cdot \ln \sqrt{1/2})}{\ln \sqrt{1/2}} = -\dfrac{W(\frac{1}{2}\ln\frac{1}{2})}{\frac{1}{2}\ln\frac{1}{2}} = \dfrac{2}{\ln 2}\ln\frac{1}{2} = -2\,, \\ -\dfrac{W(-1 \cdot \ln \sqrt{1/2})}{\ln \sqrt{1/2}} = -\dfrac{W(\frac{1}{2}\ln 2)}{\frac{1}{2}\ln\frac{1}{2}} = \dfrac{2}{\ln 2}\,W(\frac{1}{2}\ln 2) \end{cases}\,. \tag{10}$$

However, if we go back to $W(\frac{1}{2}\ln\frac{1}{2}) = W(-\frac{1}{2}\ln 2)$, we see that it is the negative range that bifurcates into $W_0(\frac{1}{2}\ln\frac{1}{2})$, which we already got, and to $W_{-1}(-\frac{1}{2}\ln 2)$.