Math Diversion Problem 401

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> Zathras is used to being beast of burden to other people's needs. Very sad life... Probably have very sad death. But, at least there is symmetry.
>
> — Zathras (a character on Babylon 5)

The YouTube video is found at:

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Source: ?
Title: ?
 Presenter: ?
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1 The Problem

Given the relation

$$x^2 = \left(\frac{1}{2}\right)^x , \tag{1}$$

find the real values of x.

Comment 1: This is such a nice problem to learn about the Lambert W function. We're going to take this one all the way down to the numbers, to learn about $W_0(\cdot)$ and $W_{-1}(\cdot)$.

Comment 2: By the way, WolframAlpha tells me that this problem has but three solutions, all real, which are

$$x = -4 , \quad -2 , \quad \approx 0.766665 . \tag{2}$$

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B , \tag{3}$$

then

$$z = W(B)\,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{5}$$

then

$$z = W_a(B)\,, \tag{6}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{7}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

Let's first rewrite (1) as

$$x = \pm\left(\sqrt{1/2}\right)^x\,, \tag{8}$$

and then convert it to the form

$$-x\left(\sqrt{1/2}\right)^{-x} = \pm 1\,. \tag{9}$$

Now, we apply the Lambert lemma above to get

$$-x = W_{\sqrt{1/2}}(\pm 1) = \frac{W(\pm 1 \cdot \ln\sqrt{1/2})}{\ln\sqrt{1/2}}\,. \tag{10}$$

Therefore,

$$x = \begin{cases} -\dfrac{W(+\ln\sqrt{1/2})}{\ln\sqrt{1/2}} = -\dfrac{2}{\ln 2}\,W(\tfrac{1}{2}\ln\tfrac{1}{2}) = \dfrac{2}{\ln 2}\,W(-\tfrac{1}{2}\ln 2)\,, \\ -\dfrac{W(-\ln\sqrt{1/2})}{\ln\sqrt{1/2}} = \dfrac{2}{\ln 2}\,W(-\tfrac{1}{2}\ln\tfrac{1}{2}) = \dfrac{2}{\ln 2}\,.W(\tfrac{1}{2}\ln 2)\,, \end{cases} \tag{11}$$

where $-\dfrac{1}{\ln\sqrt{1/2}} = \dfrac{2}{\ln 2}$. Now, since $\arg = \frac{1}{2}\ln 2 > 0$, then the Lambert W function $W(\frac{1}{2}\ln 2) = W_0(\frac{1}{2}\ln 2)$ is guaranteed both to exist and be positive. So, the x value on the bottom line of (11) is

$$x \approx 0.766665\,, \tag{12}$$

where I let WolframAlpha calculate this value. Generally speaking, except for a very few special cases in which I can look up their values in a table, I let WolframAlpha do the calculation for me.

However, when the argument (arg) of $W(\cdot)$ is negative, then the function will not have a real root unless

$$-\frac{1}{e} \leq \arg < 0\,, \tag{13}$$

or

$$-0.36787944\cdots \leq \arg < 0\,, \tag{14}$$

in which case we'll get two negative values, the greater one belonging to $W_0(\arg)$ and the lesser one (i.e., the more negative one) to $W_{-1}(\arg)$. [And, again, these values will probably be calculated by a computer.]

Now, since $\arg = -\frac{1}{2}\ln 2 \approx -0.345739$ then we need to calculate (approximate) values for $W_0(-0.345739)$ and the lesser one to $W_{-1}(-0.345739)$.

$$W_0(-0.345739) \approx -0.687769\,, \tag{15}$$

$$W_{-1}(-0.345739) \approx -1.39488\,. \tag{16}$$

So, all we've left to do now is to multiply each of these by

$$\frac{2}{\ln 2} \approx 2.88539\,. \tag{17}$$

When we do this, we get

$$\frac{2}{\ln 2} W_0(-0.345739) \approx -1.98448\,, \tag{18}$$

$$\frac{2}{\ln 2} W_{-1}(-0.345739) \approx -4.02477\,. \tag{19}$$

Now, we know that these answers are only approximate, so the fact that they are close to integer values suggests that we test the original equation for the possible solutions -2 and -4, and, in fact, these do work. So it seems that WolframAlpha was correct.

Furthermore, it might help, when first encountering such a problem, to graph it. In this case, we would graph the two functions

$$f(x) = x^2 \quad \text{and} \quad g(x) = \left(\frac{1}{2}\right)^x\,, \tag{20}$$

and then look for intersection points between them.