Math Diversion Problem 403

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The first "modern" text in algebra, van der Waerden's Modern Algebra, which appeared in 1931, was heavily influenced by Emmy Noether. It is an enlightening exercise to compare this work with algebra books of just a few decades earlier to see the profound influence that she had on our present conception of algebra. Nevertheless, even Noether realized that one needs to be familiar with a wide variety of concrete examples from all parts of mathematics before one can understand the value of the generalizations she was able to make.

— Victor J. Katz

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=TBCeQCkJC4I
Title: An Interesting Equation | Problem 301
 Presenter: aplusbi
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1 The Problem

Given the relation

$$2z\cos\theta - z^2 = 1\,, \tag{1}$$

find the values for z.

2 The Solution

Let's begin by making an observation. The complex number z is commonly parameterized in one (or both) of two ways:

$$z = a + bi\,, \tag{2}$$

$$\text{or } z = re^{i\theta}\,. \tag{3}$$

However, the Given relation (1) provides us with only one constraint on these unknowns. Hence, we shouldn't expect to be able to solve for both a, b or r, θ. Let's see what we can solve for.

I prefer to begin by taking the complex conjugate of (1):

$$2\overline{z}\cos\theta - \overline{z}^2 = 1\,. \tag{4}$$

Then, we can rewrite the both of them as

$$2z\cos\theta = z^2 + 1\,, \tag{5}$$

$$2\overline{z}\cos\theta = \overline{z}^2 + 1\,. \tag{6}$$

Now, we just divide one of these by the other, to get

$$\frac{z}{\overline{z}} = \frac{z^2+1}{\overline{z}^2+1}\,, \tag{7}$$

which can be rewritten in the form

$$z(\overline{z}^2 + 1) = \overline{z}(z^2 + 1)\,. \tag{8}$$

And this can be reformed as

$$z - \overline{z} = z\overline{z}(z - \overline{z})\,, \tag{9}$$

which has its form in components as

$$2bi = r^2(2bi)\,. \tag{10}$$

Now, assuming that $b \neq 0$, this means that $r^2 = 1$, which implies that

$$r = 1 \quad \text{and that} \quad z = e^{i\theta}\,. \tag{11}$$

And I think I'll leave it right there.