

Math Diversion Problem 404

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My [algebraic] methods are really methods of working
and thinking; this is why they have crept in
everywhere anonymously.
— Emmy Noether

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=YwE1QV393M4>
Title: An Interesting Nonstandard Equation
Presenter: SyberMath

1 The Problem

Given the relation

$$xe^{\frac{x-1}{x}} = 1, \tag{1}$$

find the values of x .

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B, \tag{2}$$

then

$$z = W(B), \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A result we'll need:

$$W_0(e) = W(1 \cdot e^1) = 1.$$

3 The Solution

The Given (1) can be rewritten as

$$e^{1/x} = xe, \quad (4)$$

which can be further rewritten to

$$\frac{1}{x}e^{1/x} = e. \quad (5)$$

On taking the Lambert W function across this equation, we get

$$\frac{1}{x} = W(e) = 1, \quad (6)$$

as indicated by the relation provided in the Preparation section. So, we can conclude that

$$x = 1. \quad (7)$$

WolframAlpha admits no solutions in the range $-\frac{1}{e} \leq x < 0$, which makes sense because for $x < 0$, $xe^{\frac{x-1}{x}}$ is never even positive.