Math Diversion Problem 405

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[Noether] taught us to think in terms of simple and general algebraic concepts—homomorphic mappings, groups and rings with operators, ideals—and not in cumbersome algebraic computations; and she thereby opened up the path to finding algebraic principles in places where such principles had been obscured by some complicated special situation.
— Pavel Alexandrov

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=0IpU9Ly_6lw
Title: An Interesting Homemade Equation
 Presenter: SyberMath
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1 The Problem

Given the relation

$$e^{1-x} = 1 - \frac{\ln x}{x}\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A result we'll need:

$W_0(-e^{-1}) = -1$.

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3 The Solution

The Given (1) can be rewritten as

$$ee^{-x} = 1 - \frac{\ln x}{x}\,, \tag{4}$$

which can be further rewritten to

$$exe^{-x} = x - \ln x = \ln e^x - \ln x = \ln\left(e^x/x\right). \tag{5}$$

Next, let $y = xe^{-1}$, then we get

$$ey = \ln y^{-1} = -\ln y\,. \tag{6}$$

So that

$$y^{-1}\ln y^{-1} = e\,. \tag{7}$$

On taking the Lambert W function across this equation, we get

$$\ln y^{-1} = W(e) = 1\,, \tag{8}$$

So, we can conclude that

$$y = \frac{1}{e}\,. \tag{9}$$

Backtracking to x:

$$xe^{-x} = \frac{1}{e}\,. \tag{10}$$

Then

$$-xe^{-x} = -\frac{1}{e}\,. \tag{11}$$

On taking the Lambert W function across this equation, we get

$$-x = W(-\frac{1}{e}) = -1\,. \tag{12}$$

And finally

$$x = 1\,. \tag{13}$$